LEARNING RESEARCH AND DEVELOPMENT CENTER

1984/38

A DEVELOPMENTAL THEORY OF NUMBER UNDERSTANDING

LAUREN B. RESNICK

University of Pittsburgh

A DEVELOPMENTAL THEORY OF NUMBER UNDERSTANDING

Lauren B. Resnick

Learning Research and Development Center
University of Pittsburgh

1984

The research reported herein was supported by grants from the National Science Foundation and the National Institute of Education, U.S. Department of Education. The opinions expressed do not necessarily reflect the position or policy of the granting agencies, and no official endorsement should be inferred.

CHAPTER 3 LAUREN B. RESNICK

A Developmental Theory of Number Understanding[1]

Research on the psychological processes involved in early school arithmetic has now cumulated sufficiently to make it possible to construct a coherent account of the changing nature of the child's understanding of number during the early school years. Earlier work, concerned largely with preschool children's informal arithmetic (e.g., Fuson & Hall, Chapter 2; Gelman & Gallistel, 1978; Ginsburg, 1977), has established the strength and the limits of the number understanding that children typically bring with them to school. My concern in this chapter will be to develop a plausible account of how number concepts are extended and elaborated as a result of formal instruction. The chapter will outline a theory of number representation for three broad periods of development: (*a*) the preschool period, during which counting and quantity comparison competencies of young children provide the main basis for inferring number representation; (*b*) the early primary period, during which children's invention of sophisticated mental computational procedures and the mastery of certain forms of story problems point to two important expansions of the number concept; and (*c*) the later

[1]The research reported here was supported by grants from the National Science Foundation and the National Institute of Education, United States Department of Education. The opinions expressed in this chapter do not necessarily reflect the position or policy of the granting agencies, and no official endorsement should be inferred.

ISBN 0-12-204780-6

primary period, during which the representation of number is modified to reflect knowledge of the decimal structure of the counting and notational system.

My account of developing number understanding is based heavily on recent work—some reported in this volume—that is providing a series of formal models of the knowledge underlying various observed arithmetic performances by children of different ages. Each of these models has been constructed to account for a particular set of performances, but there has been no systematic effort to link them into a developmental sequence. Nevertheless, an examination of the existing models strongly suggests a sequential development of mathematics competence that is characterized by (*a*) an expanding and successively elaborated set of schemata that organizes number knowledge, and (*b*) the linking of these schemata to increasingly complex procedural knowledge. In the course of the chapter I will clarify exactly what is to be understood by the terms *schematic* and *procedural* knowledge. It is important to note, however, that in stressing both procedural and schematic knowledge and their links, current theories of mathematical understanding offer promise of joining two hitherto separate and largely competing strands of research on mathematical development. These are (*a*) the behavioral, which has concentrated on number performance skills and has viewed growth in mathematical ability as the addition of successive performance skills; and (*b*) the cognitive–developmental, which has focused on changing concepts of number but has often paid little attention to the manifestation of these concepts in actual number performances.

NUMBER REPRESENTATION IN THE PRESCHOOLER: THE MENTAL NUMBER LINE

This account begins by considering what understanding of number can be assumed as the typical child enters school. Several lines of evidence point to the probability that by the time they enter school most children have already constructed a representation of number that can be appropriately characterized as a

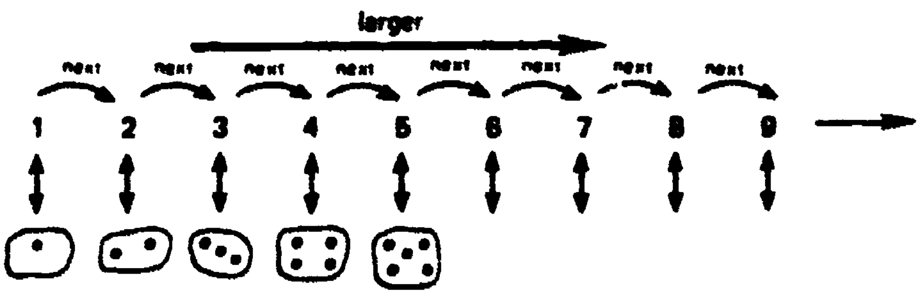


Figure 3.1 The mental number line.

mental number line. That is, numbers correspond to positions in a string, with the individual positions linked by a "successor" or "next" relationship and a directional marker on the string sp cifying that later positions on the string are larger (see Figure 3.1). This mental number line can be used both to establish quantities by the operations of counting and to directly compare quantities. By combining counting and comparison operations, a considerable amount of arithmetic problem solving can also be accomplished.

Counting

Several extensive studies of counting in preschool children provide the basis for inferring the number knowledge typical of children as they enter school. These include Gelman and Gallistel's (1978) study of counting and number concepts in 2- through 5-year-olds, and Fuson and Briars' (Fuson & Mierkiewicz, 1980) work on counting (see also Siegler & Robinson, 1982; Steffe, Thompson, & Richards, 1982). These investigators have shown that from a very early age, children can reliably count sets of objects and thus establish their cardinality. Greeno, Riley, and Gelman (1978) have developed a computational program that simulates the counting performances observed by Gelman and Gallistel and that is in good accord with the data reported by the other major investigators as well. This model provides the basis for my characterization of the mental number line.

At the core of the Greeno *et al.* model of children's counting is an ordered list of numerlogs linked by a successor (Next) relationship as shown in Figure 3.1. The program establishes the quantity of a set by a procedure that uniquely links each object in the set with one of the numerlogs and then designates the last numerlog named as the number in the set. The figure shows direct links between the smallest numerlogs and patterned set displays. These links represent the kind of knowledge that would allow children to *subitize* very small sets—that is, to quickly provide the appropriate number name without actually counting—through direct pattern recognition. This ability has been attributed to children as young as 3 or 4 by Klahr and Wallace (1976), although Greeno *et al.* argue that the appearance of subitizing may be a function of the rapid perceptual grouping of small sets as part of the counting process rather than as a separate means of quantifying an array. Without attempting to decide between these two accounts of rapid quantification of small sets, it seems reasonable to propose that it is through extensive practice with counting as a method of establishing quantity that the numerlog list is gradually transformed from a string of words into a *representation of quantity* in which each position (number name) in the list comes to stand for a quantity. Recent work by Comiti (1980) has shown that the counting list and its use in determining quantity is established only for relatively

small numbers by the time a child enters school. For quantities in the teens and twenties, many 6-year-olds are unreliable counters and are not able to use counting to establish equivalence of sets—something they can do at a much younger age for smaller set sizes. In addition, children have difficulty for some time in starting a count at a number other than 1, indicating that individual successor links are not fully established for some parts of the string (Fuson, Richards, & Briars, 1982). It is thus clear that the number representation shown in Figure 3.1 is still developing for larger quantities once school begins.

Quantity Comparisons

A smaller but still significant body of work on magnitude comparison by children allows us to further specify the characteristics of the mental number line as the child enters school. Typically in magnitude comparison tasks, two "target" numbers are named and the subject asked to decide which is larger or "shows more." Variations of this task have been extensively used with adults (e.g., Potts, Banks, Kosslyn, Moyer, Riley, & Smith, 1979). Investigators studying children (Schaeffer, Eggleston, & Scott, 1974; Sekuler & Mierkiewicz, 1977; Siegler & Robinson, 1982) have established that children can perform this task accurately by the age of 5 or earlier—at least for small numbers.

What additions to the mental number line are necessary to account for this ability? If we were to add to the quantity representation already described a directional coding that specified that later numbers in the string represented larger quantities, a child could compare two named numbers by starting up the string from 1, noting when the first of the two target numbers was reached and then labeling the *other* number as "more" or "larger."

Although this is logically possible, it seems psychologically unlikely for at least two reasons. First, it forces the child to treat *more* as if it were the marked item in the "more–less" pair. A number of investigators, beginning with Donaldson and Balfour (1968), have demonstrated that *more* is unmarked—that is, it is more easily learned and more quickly accessed than *less*. Second and even more compelling, 5-year-old children, like adults, show a characteristic pattern of reaction times for these comparison tasks: They take *longer* to make comparison judgments the *closer* the two target numbers are. If a child were using the counting-up strategy to make comparisons, the time to make a mental magnitude comparison should be a function of the size of the smaller number and not of the size of the split between the two numbers. The existence of the split effect suggests that the child's number representation has important analog features that allow direct comparison of number positions. It is as if perceptual comparisons of positions on a measuring stick were being made: when positions are closer together, it takes longer to discriminate between them than when they are further apart.

Because of the split effect for number comparisons, we can attribute to children entering school two other features of the mental number line: (*a*) a directional marker on the line that interprets positions further along the line as "larger" (as shown in Figure 3.1), and (*b*) an ability to directly enter the positional representation for a number upon hearing its name (i.e., without counting up to it). Both of these features play a role in various kinds of informal arithmetic performances that have been observed in preschool children.

Informal Arithmetic

As just noted, the mental number line can be used both to establish quantities by the operations of counting and to directly compare quantities. By combining counting and comparison operations, the child can also accomplish a considerable amount of arithmetic problem solving. For example, Gelman (1972), in her "magic" experiments, showed that young children could recognize when the number of items in a small set had been changed while the set was hidden from view. This would involve counting the set twice, before and after the change, and then comparing the two numbers by entering them on the mental number line. Gelman and Gallistel (1978) also document some young children's ability to "fix" a set so that it has a named quantity. A child with only the number knowledge sketched thus far could build a larger set (e.g., "fix" a set of three so it has five) by counting the three objects in the presented set and then adding in more objects by "counting on" up to five. To reduce a set (e.g., "fix" a set of five so it has three), the child would have to count the objects of the set up to three and then discard the remainder. The more efficient procedure of determining in advance that two items must be added to or deleted from the set would not yet be available to the child at this stage in the development of quantity representation.

This is not to say that the child has no resources for solving addition and subtraction problems. Ginsburg (1977) has reported a variety of successful arithmetic calculation procedures employed by preschool children, all apparently invented by the children and virtually all based on counting. An example is addition by constructing sets (on fingers or with objects) to match each addend, then counting up the combined sets. A typical procedure for subtraction—one that requires no more complicated quantity representation than the one considered thus far—is to (*a*) count out a set to match the larger number (the minuend), (*b*) count out from this set the number of objects specified in the smaller number (the subtrahend), and then (*c*) count the objects remaining in the original set.

Several investigators (e.g., Carpenter & Moser, 1982; Lindvall & Gibbons-Ibarra, 1980) have shown that young children are able to solve certain classes of story problems using counting procedures. Typically in these solutions they use only forward counting, by ones, of actual countable objects. However, some

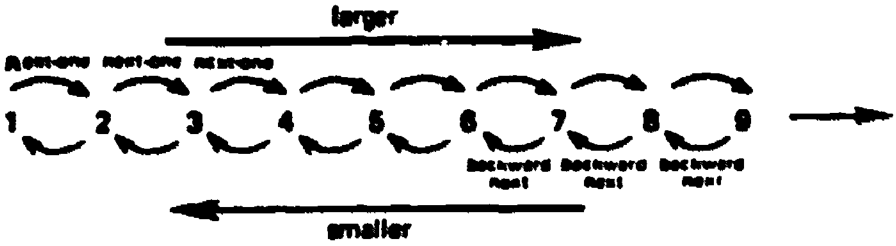


Figure 3.2 The mental number line with backward markers.

children apparently acquire the ability to use decrementing (counting backward) procedures before they enter school. This means that "backward–next" links must have been attached to adjacent numbers in their mental number line and a "smaller" (less) directional marker attached to the line as a whole (see Figure 3.2). Performances that call on backward counting include doing subtraction by counting down from the larger number. Although these performances are often used to argue that children already know important concepts of mathematics before school begins, in fact such performances require only a primitive representation of number compared to what will develop subsequently.

EARLY SCHOOL ARITHMETIC: THE PART–WHOLE SCHEMA

As long as the number line alone is used, there is no way to relate quantities to one another except as larger or smaller, further along or further back in the line. Although quantities can be compared for relative size, no precision in the relative size relationship is possible except as a specification of the number of numerlogs that must be traversed between positions in the line. Probably the major conceptual achievement of the early school years is the interpretation of numbers in terms of part and whole relationships. With the application of a Part–Whole schema to quantity, it becomes possible for children to think about numbers as compositions of other numbers. This enrichment of number understanding permits forms of mathematical problem solving and interpretation that are not available to younger children.

Figure 3.3 sketches a Part–Whole schema that plays a role in several models of children's developing number understanding (Briars & Larkin, 1981; Resnick, Greeno, & Rowland, 1980; Riley, Greeno, & Heller, Chapter 4). The schema specifies that any quantity (the whole) can be partitioned (into the parts) as long as the combined parts neither exceed nor fall short of the whole. By implication, the parts make up or are included in the whole. The Part–Whole schema thus provides an interpretation of number that is quite similar to Piaget's (1941/1965) definition of an operational number concept. To function as a tool in problem solving, the part–whole knowledge *structure* must be tied to *procedures*

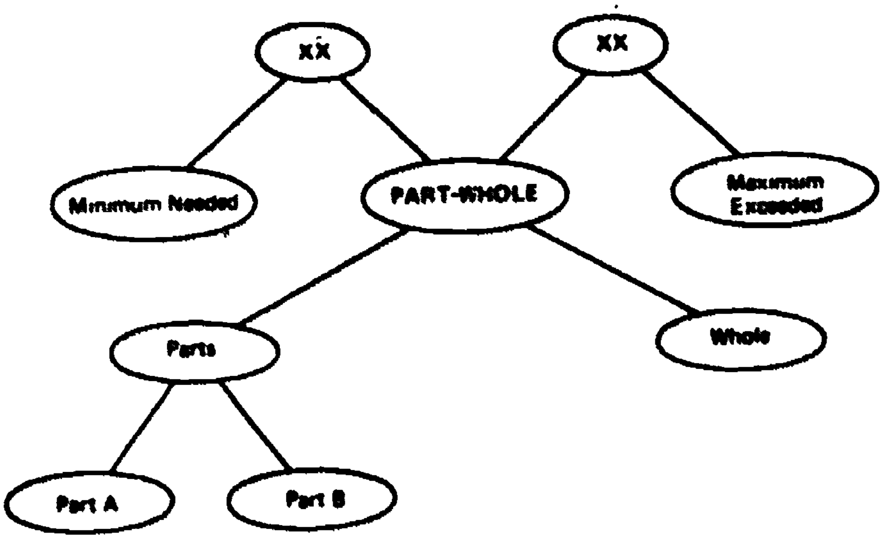


Figure 3.3 The Part–Whole schema

for constructing or evaluating quantities. The Maximum Exceeded and Minimum Needed nodes in Figure 3.3 are connected to procedures by which deletions or additions can be made to satisfy the constraint that the sum of the parts is equivalent to the whole. For example, if the numbers in the Whole and Part A slots are known, a counting-up procedure (accessed through the Minimum Needed node) can be used to fill Part B with the number needed to keep the combined parts equal to the whole.

Story Problems

The Part–Whole schema specifies relationships among triples of numbers. In the triple 2-5-7, for example, 7 is always the whole; 5 and 2 are always the parts. Together, 5 and 2 satisfy the equivalence constraint for the whole: 7. The relationship among 2, 5, and 7 holds whether the problem is given as $5 + 2 = ?$, $7 - 5 = ?$, $7 - 2 = ?$, $2 + ? = 7$, or $? + 5 = 7$. Each of these number sentences expressing the relations among the triple 2-5-7 has one or more corresponding expressions in real-world relationships or in story problems. Figure 3.4 shows how the fundamental part–whole relationship underlies several classes of story problems as well as number sentences. In each problem the whole is coded as a dot-filled bar, whether it is a given quantity or the unknown. Similarly, each part is uniquely coded. The relationship between parts and whole for all the problems, including the number sentences, is shown in the center display. Any bar can be omitted and thus become the unknown. Although number sentences and the given words of story problems cannot be mapped directly onto one another (Nesher & Teubal, 1975), each can be mapped directly onto a more abstract part–whole representation, such as the bars shown here. The Part–Whole sche-

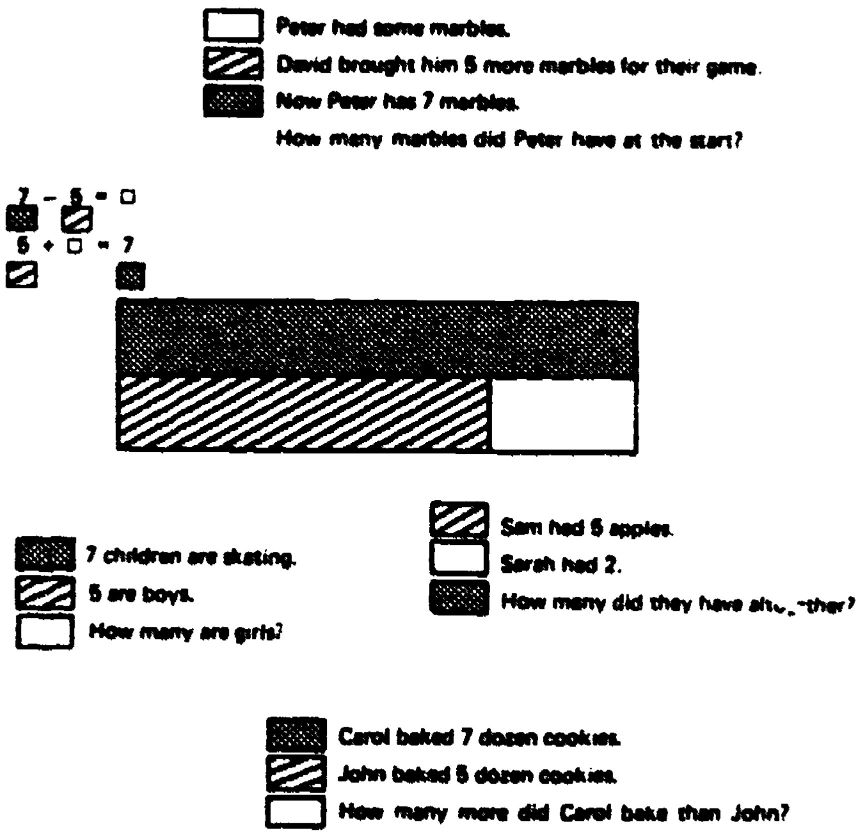


Figure 3.4 Mapping of stories and number sentences to a concrete model of Part–Whole.

ma thus provides an interpretive structure than can permit the child to either solve certain more difficult problems directly by the methods of informal arithmetic, or to convert them into number sentences that can then be solved through procedures taught in school.

Riley, Greeno, and Heller (Chapter 4) have developed a family of computational models that account for the development of competence in solving one-step addition and subtraction story problems of the kind studied by a number of investigators (e.g., Carpenter & Moser, 1982; Nesher, 1982; Vergnaud, 1982). These models suggest that it is application of the Part–Whole schema that makes it possible to solve difficult classes of story problems that children usually cannot solve until the second or third school year. These include set-change problems with the starting set unknown (e.g., *John had some marbles. Michael gave him 4 more. Now he has 7. How many did he have to start?*) and various kinds of comparison problems (e.g., *John has 4 marbles. Michael has 7. How many more does Michael have than John?*). An alternative story problem model by Briars and Larkin (1981) solves some of the more difficult problems by constructing a mental script that reflects real-world knowledge about combining and separating

objects, rather than abstract part–whole relationships. The script describes the actions in the story and allows the system to keep track of the sets and subsets involved. Yet in Briars and Larkin's model, too, it proves possible to solve unknown-first problems only by instantiating a Part–Whole schema. Both theories, then, assume that story problem solution—at least for the most difficult problems—proceeds by mapping the statements in the problem into the slots of the Part–Whole schema. This allows the numbers in the problem to be assigned to either "part" or "whole" status and permits a clear identification of whether the unknown is a part or a whole. This in turn allows flexible computational strategies, including *either* direct counting solutions (for example, by counting up from Part A if Part B must be found) *or* the construction of an appropriate number sentence and then solution of the arithmetic problem specified in the number sentence.

Mental Addition and Subtraction

We have seen that preschool children using mainly forward counting procedures are capable of solving a surprising variety of arithmetic problems as long as they have actual countable objects to aid in the calculation. During the early years of school, children come to be able to solve many of the simpler arithmetic problems "in their heads"—that is, without any overt counting. It had long been assumed that when children ceased overt counting, they had switched to an adult-like performance in which the number facts (e.g., single-digit addition or subtraction problems) were simply associations, memorized and then recalled on demand. Presumably, no reasoning went on in arriving at an answer. Recent work, however, has established quite clearly that there is an intermediate period of several years during which arithmetic problems are solved by mental counting processes. These procedures appear to be children's own inventions. There is reason to believe that the Part–Whole schema plays a role in establishing these procedures, although there is no formal theory nor very direct evidence yet available to specify that role.

Research by Groen and Parkman (1972) is the point of reference for work on simple mental calculation. Working with simple addition (two addends with sums less than 10), Groen and Parkman tested a family of process models for single-digit addition. Figure 3.5 shows the general model schematically. All of the models assumed a "counter in the head" that could be set initially at any number, then incremented a given number of times and finally "read out." The specific models differed in where the counter was set initially and in the number of increments-by-one required to calculate the sum. For example, the counter can be set initially at zero, the first addend added in by increments of one, and then the second addend added by increments of one. If we assume that each increment

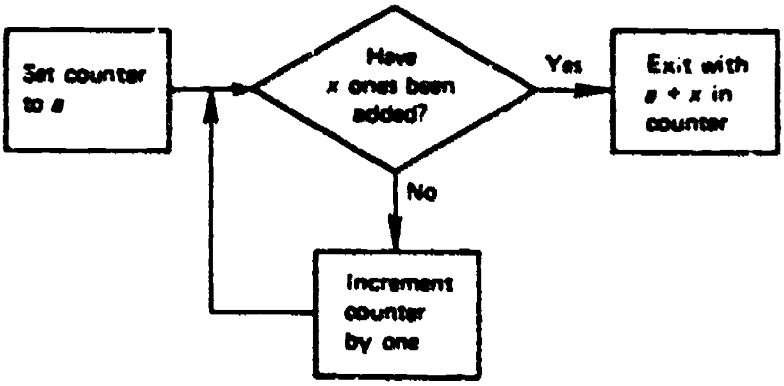


Figure 3.5 Counting model for simple addition. (From Groen & Parkman, 1972. Copyright 1972 by the American Psychological Association. Reprinted by permission.)

needs about the same amount of time to count, then someone doing mental calculation this way ought to show a pattern of reaction times in which time varies as a function of the sum of the two addends. This has become known as the *sum* model of mental addition. A somewhat more efficient procedure begins by setting the counter at the first addend and then counting in the second addend by increments of one. In this case—assuming that the time for setting the counter is the same regardless of where it is set—reaction times would be a function of the quantity of the second addend. A still more efficient procedure starts by setting the counter at the larger of the two addends, regardless of whether it is the first or the second, and then incrementing by the smaller. Obviously, this would require fewer increments. Such a procedure would produce reaction times as a function of the size of the minimum addend and has thus become known as the *min* model.

Groen and Parkman evaluated these (along with some other logically possible but psychologically implausible) models by regressing observed on predicted patterns of reaction times for each model. The finding was that children as young as first-graders used the *min* procedure. Subsequently, the *min* model has been confirmed in studies that have extended the range of problems up to sums of 18, and the ages of children from 4½ or so up to 9 or 10 (Groen & Resnick, 1977; Svenson & Broquist, 1975; Svenson & Hedenborg, 1979; Svenson, Hedenborg, & Lingman, 1976). Figure 3.6 shows a characteristic data plot. Note that problems with a minimum addend of 4 cluster together and take longer than problems with a minimum addend of 3, and so on. It is also typical that doubles (e.g., 2 + 2) do not fall on the regression line but instead are solved particularly fast. We can infer that some process other than counting is used in responding to doubles problems, a point I shall return to later.

Counting models have also been applied to other simple arithmetic tasks, especially subtraction (Svenson *et al.*, 1976; Woods, Resnick, & Groen, 1975), and addition with one of the addends unknown (Groen & Poll, 1973). In the case

of subtraction, at least three mental counting procedures are mathematically correct. One procedure would involve initializing the counter in the head at the larger number (the minuend) and then decrementing by one as many times as indicated by the smaller number (the subtrahend). In this *decrementing* model, reaction times would be a function of the smaller number. A second procedure would involve initializing the counter at the smaller of the two numbers and incrementing it until the larger number is reached. The number of increments then would be read as the answer. Reaction times for this *incrementing* model would be a function of the remainder—the number representing the difference between the minuend and subtrahend. A particularly efficient procedure would involve using *either* the decrementing *or* the incrementing process for subtraction, depending upon which required fewer steps on the counter. Reaction times would be a function of the smaller of the subtrahend and the remainder. This *choice* model is what most primary school children use, although a few second-graders use the straight *decrementing* model (See Figure 3.7). Here again, note how the doubles fall below the regression line, suggesting a faster, noncounting solution method.

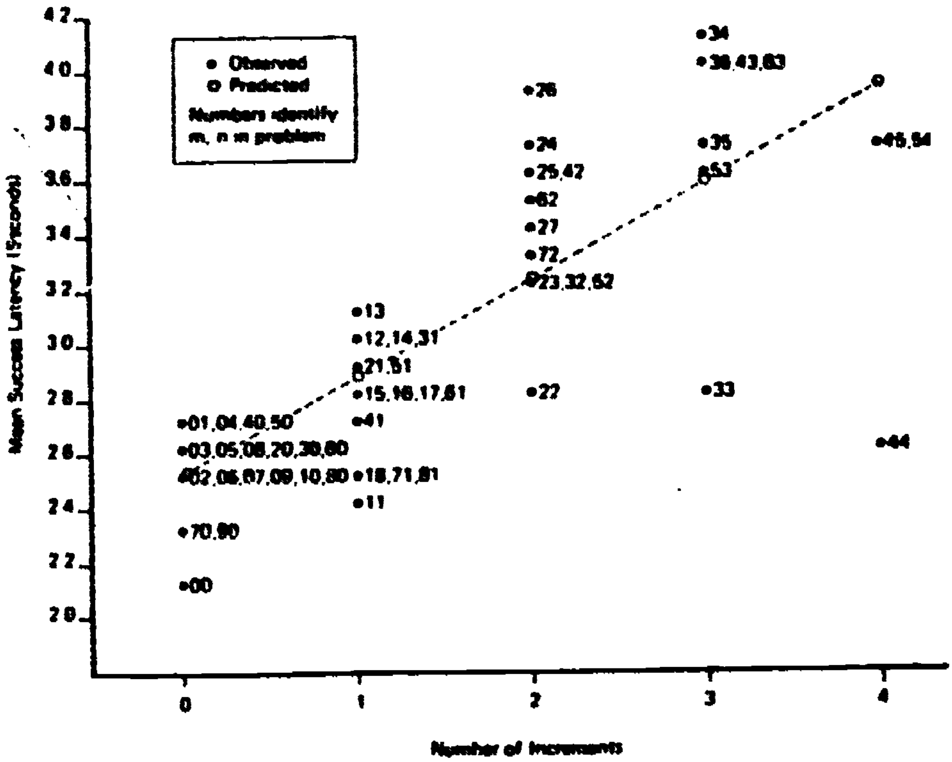


Figure 3.6 Reaction times for first graders solving addition problems. Pairs of numbers listed above or below dots stand for single-digit addition problems (e.g., 0 + 0, 0 + 1, 1 + 0). Dots indicate the average reaction times for adding each pair of numbers. (From Groen & Parkman, 1972. Copyright 1972 by the American Psychological Association. Reprinted by permission.)

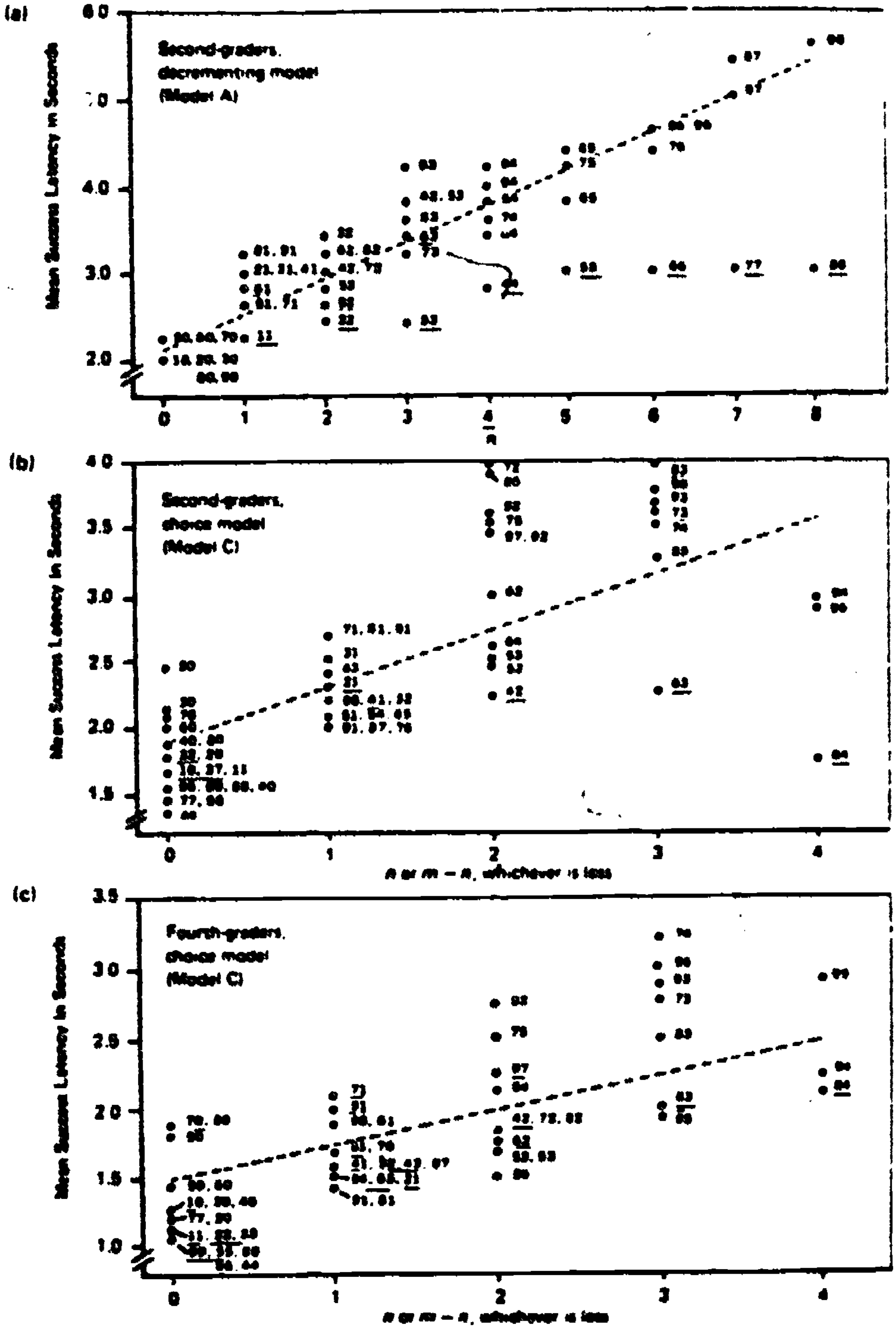


Figure 3.7 Mean reaction time patterns for three groups of children using a *decrementing* model or a *choice* model of subtraction. (Adapted from Woods *et al.*, 1975. Copyright 1975 by the American Psychological Association. Reprinted by permission.)

It is risky to attribute complex processes such as *min* and *choice* to people entirely on the basis of reaction time patterns. For this reason, it is important to ask what converging evidence exists that points to the reality of mental counting procedures. Observations of overt counting-on strategies for addition by several investigators (Carpenter, Hiebert, & Moser, 1981; Fuson, 1982; Houlihan & Ginsburg, 1981; Steffe *et al.*, 1982) suggest that the counting presumed in these models is real. Furthermore, Svenson and Broqvist (1975) interviewed their subjects after each timed trial and found that on about half of the problems, children reported counting up from the larger number (by ones or in larger units). Finally, evidence comes from comparing children's reaction-time patterns for addition with those of adults, whom we can assume retrieve elementary addition and subtraction facts by some kind of direct "look-up" procedure. Adults show much faster reaction times and a far shallower slope (20 msec) when their data are fit to *min* than do children (Groen & Parkman, 1972). Their slope, which is presumably the time needed for each count, seems too fast to represent anything like a real counting procedure. Groen and Parkman suggested that this shallow slope might be an artifact of averaging over many trials in which the answers were looked up (presumably producing a flat slope) and a few trials in which they were counted. More recently, Ashcraft and Battaglia (1978) have suggested that adults do not produce a linear increase in time as the minimum addend grows, but instead produce a positively accelerating curve that is best fit by *square of the sum*. Ashcraft and Fierman (1982) tried to fit children's data to *sum*^{2}, but not until fourth grade did *sum*^{2} provide the best fit. Younger children thus do appear to solve addition problems by counting. The converging evidence for subtraction is less rich, although some of Svenson's (Svenson & Hedenborg, 1979; Svenson *et al.*, 1976) subjects described the *choice* strategy in interviews.

It is important to note, however, that while *min* and *choice* appear to be the dominant procedures during the early school years, they are not the only ones used. Several investigators have noted the use of special shortcut mental addition strategies by children during this period. These have been documented in some detail by investigators (Carpenter & Moser, 1982; Houlihan & Ginsburg, 1981; Svenson & Hedenborg, 1979; Svenson & Sjoberg, in press) who used verbal protocols and reaction times to document strategies that made special use of addition and subtraction facts that children had committed to memory and could retrieve directly. Most common were the use of solutions with tie references (e.g., 3 + 4 is solved by saying 3 plus 3 is 6, plus 1 more makes 7; or 13 − 6 is solved by saying 12 minus 6 is 6, plus 1 is 7). Saxe and Posner (Chapter 7) found similar strategies among illiterate Africans. Less frequent, but of considerable interest because they signal a developing appreciation of the decimal number system, are solutions that depend on knowledge of tens complements. For example, 6 + 5 is converted to 6 + 4 (= 10), plus 1 more. Or, for subtraction, 11 − 4 is converted to 10 − 3 + 1. These shortcut procedures provide evidence that

children understand the compositional structure of numbers and are able to partition and recombine quantities with some flexibility.

The Origins of Invented Arithmetic Procedures

What must be added to the mental number line representation to account for the predominance of *min* and *choice* and for the occurrence of special tie- and complements-referenced strategies during the earliest school years? In considering this question, we should keep in mind that these strategies are not directly taught in most school programs. Extensive practice in addition and subtraction is given, some of it organized to highlight commutative pairs in addition and the complementarity of addition and subtraction. But the actual counting procedures and the conversions to make use of tie and complements facts must usually be invented by the children themselves—sometimes in the face of strictures against overt counting. Indeed, the invented character of *min* has been demonstrated directly (Groen & Resnick, 1977). We taught preschool and kindergarten children a procedure for addition that involved counting out both sets. Half of the children switched to *min* without further instruction after about 12 weeks of practice sessions.

The invented character of *min* and *choice* poses an interpretive challenge, for neither of these procedures appears to derive in a straightforward, mechanical way from the overt counting procedures observed among younger children. That is, they are not simply shortcuts, in the sense of dropping redundant steps. Indeed, in each case a new step—deciding which number to start counting from—is added. Furthermore, *min* seems to depend upon the mathematical principle of commutativity, the recognition that the sum of two numbers is the same regardless of the order in which they are added, and *choice* appears to depend upon recognition of the complementarity of addition and subtraction. Yet neither of these principles is directly taught to children in the earliest grades of school any more than the actual *min* and *choice* procedures are taught, and no study has suggested that children who use them have any verbal awareness of the general principles involved. Our interpretive task, then, is to account for the emergence of *min* and *choice* as procedures that accord with mathematical principles of commutativity and complementarity but are not systematically derived from those principles. There are several possible explanations to consider.

A "PAIR-EQUIVALENCE" ACCOUNT

The simplest account of the discovery of *min* would assert that the special relationships between certain pairs of problems (e.g., 3 + 4 and 4 + 3; 2 + 7 and

7 + 2) are noticed after extensive practice on the individual pairs, through a general learning process that looks for regularities and shortcuts after a procedure becomes at least partially automated (cf. Anderson, 1981; Klahr & Wallace, 1976). In this view, the child would notice that specific pairs of problems yielded the same answer and would infer that they could be substituted for one another. A preference for efficiency would then lead to the strategy of always starting the count at the larger number.

This seems plausible until we consider that if the child is to notice the equivalence of two problems, the result of both pairs must be present in short-term memory simultaneously so that they can be compared. This could happen in two ways. First, if commuted pairs (e.g., 7 + 3 and 3 + 7) were presented successively, the result of the first calculation might still be present when the second calculation was completed. However, in our experiment (Groen & Resnick, 1977) the children invented *min* under controlled practice conditions in which these pairings of problems did not occur. Practice on paired problems, then, cannot be a general explanation for the development of *min*, although it may play a role in some cases. A second possibility is that the result of 7 + 3 can be quickly retrieved when 3 + 7 is computed. But this would mean that 7 + 3 was already known as a retrievable addition fact. If such retrievable facts were available, however, children would not need to use counting procedures to compute the answers to simple addition and subtraction problems. It therefore appears implausible to attribute the discovery of *min* to simply noticing the common outcome of different orders of performing addition.

A modified version of the pair-equivalence account may survive, however. This version would assume that the equivalence was noticed first for very easily computable pairs (e.g., those involving an addend of 1). It seems plausible that the sum of 7 + 1 could be retrieved (or constructed) fast enough to be simultaneously present in short-term memory with the sum of 1 + 7. Having noted equivalence for a subset of the addition pairs, a child might plausibly construct a more general commutativity rule that could be applied to other pairs.

A "DEFAULT" ACCOUNT

Another possibility is that children begin by *assuming* that arithmetic operations are commutative and only gradually learn that some (for example, subtraction) are not. This would lead them to try *min* procedures in the search for less-effort processes. Since *min* "works" (i.e., the answer turns out to be correct when checked by counting the whole joint set, and adults do not comment on the result as wrong), they would retain it as the preferred procedure. In support of this possibility is the observation that children frequently attempt to commute subtraction problems. That is, when given the problem 2 − 5, they respond with

3 rather than −3, 0, or "you can't do it"—any of which would indicate recognition of the noncommutativity of subtraction. Another common attempt to commute in subtraction is shown by giving solutions such as:

$$\begin{array}{r} 348 \\ -\underline{169} \\ 221 \end{array}$$

A child would arrive at this incorrect answer by "subtracting within columns" (Brown & Burton, 1978)—that is, by taking the smaller number from the larger in each column regardless of which is on top.

The Gelman and Gallistel (1978) analysis of young children's counting makes it clear that they proceed in accord with an "order-invariance" principle—that is, they recognize that *objects* can be counted in any order, although the numerlogs must be assigned in their standard sequence. A natural extension of order-invariance would allow subsets as well as individual objects to be enumerated in any order. This would allow *min* to emerge as part of a general search for low-effort solutions without requiring that the child construct any kind of commutativity rule.

Neches (1981, and personal communication) has provided a formal account of how *min* might be discovered on such a "default" basis. His computer model of addition begins by performing a *sum* solution in which both subsets are counted out and the combined set recounted. After a number of practice trials, the system notices that a *portion* of the counting process for finding the total is redundant with the original counting process for each of the subsets. In recounting for the problem 2 + 5, for example, the first two counts are redundant with counting out the first subset, and the first five counts are redundant with the original count for the second subset. The system has some general redundancy elimination mechanisms that lead it to reuse existing computations rather than duplicate them. This means that two counting-on solutions are constructed, one for each addend. The system eventually comes to count on from the *larger* addend (thus performing the *min* procedure) because it can detect a redundancy when the smaller-addend alternative is tried.

A "PART-WHOLE" ACCOUNT

Still another possibility for the emergence of *min* is that children apply a simple Part–Whole schema to addition. For example, a child could solve addition problems by binding the given addends to the Part slots of the schema. Since the slots contain no order information, the addends can now be used in either order to discover the value of the Whole. This is an attractive explanation of *min* because it also accounts economically for the discovery of *choice*. Part–Whole puts the three terms of a complementary addition–subtraction pair into a stable

relationship with one another. For the problem 9 − 7, for example, 9 would fill the Whole slot and 7 one of the Part slots. For 9 − 2, 9 would fill the Whole slot and 2 one of the Part slots. In finding the missing part (using the procedures attached to the Minimum Needed and Maximum Exceeded nodes of the schema), the child would become aware of the complementary relationships between 9 − 2 = 7 and 7 + 2 = 9. This complementary relationship could then be used to generate least-effort solution rules. Part–Whole also provides a convenient account of the basis for complement- and tie-based shortcut procedures.

Application of Part–Whole seems to be a plausible account for the emergence of *min* and *choice*, at least to the extent that it is plausible to attribute the Part–Whole schema to children at an early enough age so that it precedes *min* and *choice* as part of the knowledge structure. We have mixed evidence here. On the one hand, a fully general Part–Whole schema does not seem to be reliable until the age of 7 or 8. This is when children master Piagetian class inclusion problems (Inhelder & Piaget, 1964/1969), which are part–whole problems without a requirement of specific numerical quantification. It is also the age at which children can reliably solve those story problems that clearly depend on the part–whole structure (e.g., set-change problems with the starting set unknown). This age would be too late to account for *min*, although it is possibly an acceptable age for *choice*, which as far as we know develops later.

Still, several investigations point to an earlier understanding of certain class relationships than the Piagetian studies have suggested. For example, Markman and Siebert (1976) have shown that if the class character of the Whole set is emphasized by the wording of the problem, children can perform class inclusion problems quite early, and Smith and Kemler (1978) have shown that kindergarten children use component dimensions in certain kinds of classification tasks. Furthermore, children as early as first grade can solve comparison story problems when they are worded so as to make the part–whole relations evident (see Riley *et al.*, Chapter 4). Thus, it seems plausible that children may possess at least a simple version of the Part–Whole schema at a quite young age but may not yet have learned all of the situations where it is appropriate to apply it. Addition and subtraction of small numbers, unencumbered by story content, may be one of the easy-to-recognize situations. Indeed, application of a primitive Part–Whole schema to simple number problems may be an important step in developing a more elaborate version, including many procedural connections, that will play a role in subsequent development of number knowledge.

DEVELOPMENT OF DECIMAL NUMBER KNOWLEDGE

All of the research discussed so far has focused on small numbers—quantities up to about 20. From this work we are able to trace a probable course of

development of number representation in which the fundamental relationships between numbers are units. Yet the introduction of decimal numbers, which form an important part of the primary school mathematics curriculum, demands that a new relationship among numbers be learned. This relationship is based on tens rather than units. The initial introduction of the decimal system and the positional notation system based on it is, by common agreement of educators, the most difficult and important instructional task in mathematics in the early school years. Starting in about second grade, most schools begin to teach children about the structure of two-digit numbers. Toward the end of second grade, addition (and in some schools, subtraction) with regrouping is introduced. What is known about the development of knowledge of the base ten system—its representation in written form, and the calculation algorithms that are based on it? How does the quantity representation change as skill in the posititional notation system develops? These questions are addressed below.

Numbers as Compositions of Tens and Units: Restriction and Elaboration of the Part–Whole Schema

We have already seen that an important aspect of the development of number during the early school years is the interpretation of numbers as compositions of other numbers—that is, the application of the Part–Whole schema to numbers previously defined solely in terms of position in a linear string. In story problems and simple mental arithmetic, the Part–Whole schema is applied with few restrictions and little elaboration. I will now try to show that the development of decimal number knowledge can be understood as the successive elaboration of the Part–Whole schema for numbers, so that numbers come to be interpreted by children as compositions of units and tens (and later of hundreds, thousands, etc.) and are seen as subject to special regroupings under control of the Part–Whole schema.

There is far less research to draw on in making this characterization of developing place value knowledge than there is for early number concepts, story problems, and simple arithmetic. In addition to ongoing work in our own laboratory, I will refer to empirical and theoretical work by several others in building this account of stages of development in decimal number understanding. The account must be viewed as tentative and subject to modification as further evidence on the development of understanding of the decimal number system accumulates. In particular, the later stages of this account are based on data from a small number of children who were receiving remedial instruction in our laboratory. We need to extend this data base to include more children—especially those children who acquire place value understanding without the special intervention included in our studies.

We can identify three main stages in the development of decimal knowledge. First, there is an initial stage in which a unique partitioning into units and tens (e.g., 47 is 4 tens plus 7 units) is recognized. Next, in stage two, children recognize the possibility of multiple partitionings of a quantity. This second stage occurs in two phases: Multiple partitionings are (*a*) arrived at empirically (e.g., the equivalence of 30 tens plus 17 units to 40 tens plus 7 units is established by counting), and (*b*) established directly by application of exchanges that maintain equivalence of the whole (e.g., 40 + 7 = 30 + 17, because 1 ten can be exchanged for 10 units). Third, a formal arithmetic stage appears in which exchange principles are applied to written numbers to produce a rationale for algorithms involving carrying and borrowing.

Stage One: Unique Partitioning of Multidigit Numbers

The earliest stage of decimal knowledge can be thought of as an elaboration of the number line representation so that, rather than a single mental number line linked by the simple "next" relationship, there are now two coordinated lines, as sketched in Figure 3.8. Along the rows a "next-by-one" relationship links the numbers. This can be extended indefinitely, as shown in the top row, indicating that a units representation of number coexists with a decimal representation. Along the columns a "next-by-ten" relationship links the numbers. In a fully developed number representation this "next-by-ten" link might hold for the numbers inside the matrix as well as for those along the edges, permitting more efficient addition or subtraction of the quantity 10 than of other quantities. Earlier, and perhaps indefinitely, the "inside" links (e.g., 37 + 10 = 47) might be constructed on each occasion of use by a procedure that decomposes the two-digit number into a tens and a units portion (37 = 30 + 7), then adds 10 to the tens portion (30 + 10 = 40), and finally adds back the units (40 + 7 = 47). In either case, the most important feature of this new stage of number understanding is that each of the numbers is represented as a *composition* of a tens value and a units value. This means, in effect, that two-digit numbers are interpreted in terms of the Part–Whole schema, with the special restriction that one of the parts be a multiple of 10.

There is some evidence that this compositional structure of the numbers arises first in the context of oral counting—that is, that it is not at first tightly linked to quantification of large sets of objects or to grouping of units by tens. Several investigators (Fuson *et al.*, 1982; Siegler & Robinson, 1982) found that many 4- and 5-year-olds could count orally well into the decades above 20 and that their counting showed evidence of being organized around the decade structure. For example, the most common stopping points in the children's counting were at a number ending in 9 or 0 (e.g., 29 or 40); and their omissions in the

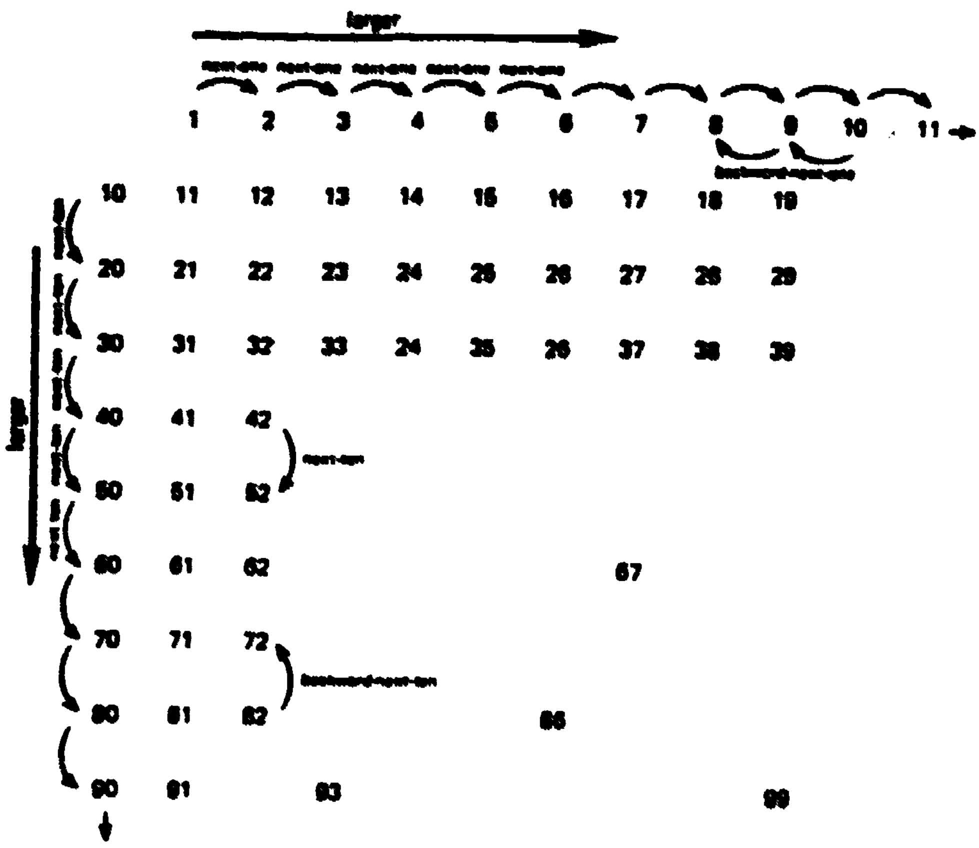


Figure 3.8 Earliest stage of decimal knowledge represented as two coordinated mental number lines.

number string tended to be omissions of entire decades (e.g., ". . . 27, 28, 29, 50 . . ."). They also sometimes repeated entire decades (e.g., " . . . 38, 39, 20, 21 . . .") and sometimes made up nonstandard number names reflecting a concatenation of the tens and the units counting strings (e.g., " . . . twenty-nine, twenty-ten, twenty-eleven . . ."). Finally, these children could usually succeed in counting on within a decade higher than their own highest stopping point when asked by the experimenter to start counting from a particular number, such as 51 or 71.

In our own work on place value, we have collected many observations of primary school children's methods of establishing the quantity shown in displays of blocks or other objects coded for decimal value (see Figure 3.9 for examples of such displays). The typical method that children use in this kind of task is to begin with the largest denomination and enumerate the blocks of that denomination using the appropriate counting string (e.g., 100, 200, 300, etc., for hundreds blocks), then add in successive denominations by counting on using the appropriate counting string. A successful quantification of the display in Figure 3.9a, for example, would produce the counting string: 100, 200, 300, 400, 410, 420, 430,

440, 450, 460, 461, 462, 463. A few children, mainly those who show the most sophisticated knowledge of other aspects of place value, count all denominations by ones and then "multiply" by the appropriate value (e.g., for Figure 3.9a: 1, 2, 3, 4, 400; 1, 2, 3, 4, 5, 6, 460; 1, 2, 3, 463). However, counting using the decimally structured number strings seems to be the earliest application of decimal knowledge to the task of quantifying sets. Furthermore, between simple oral counting competence and the successful use of the decimal-structured counting strings for quantification, there seems to be a period during which the child knows the individual strings well enough to use them separately for quantifica-

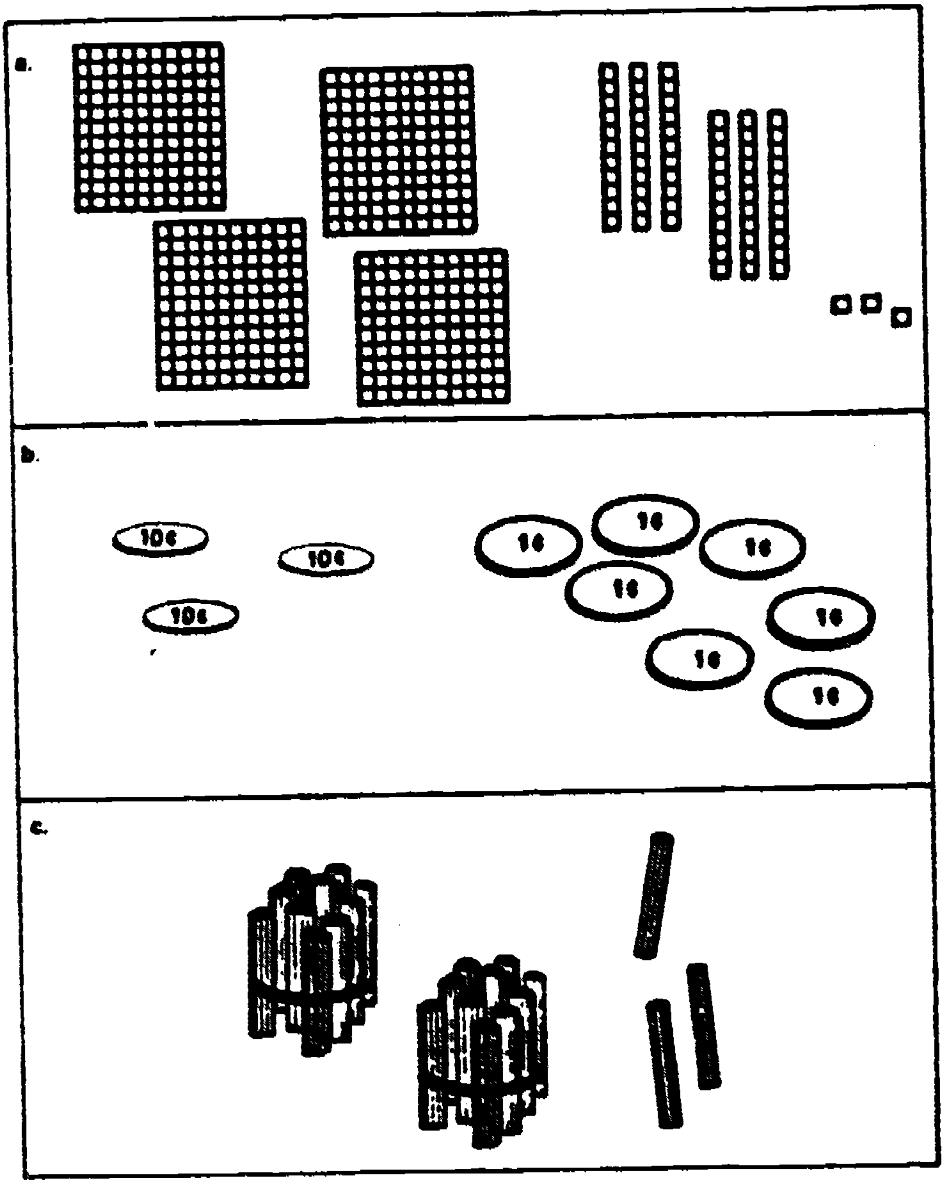


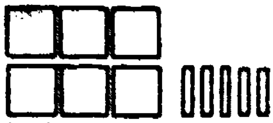
Figure 3.9 Examples of displays used in research on decimal knowledge

tion but cannot coordinate the use of several strings within a single quantification task. In one of our studies, for example, all of the third-grade children we interviewed could count any single block denomination, but more than half of the children became confused when two or more denominations were to be quantified. Examples from the protocols of two such children appear in Figure 3.10.

Other performances characteristic of children in this early stage of decimal number knowledge suggest that children typically recognize the relative values

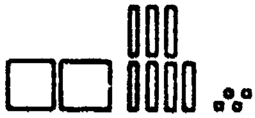
a) Alice

E. Shows:



S. (Touching the hundreds) 100, 200, 300, 400, 500, 600 (touching the tens) 7, 8, 9, 10, 11 . . . 611.

E. Let's try one more like this. How about this one?



S. (Touching the hundreds) 100, 200 . . (touching the tens) 201, 202, 203, 204, 205, 206, 207 . . . (touching the ones) 208, 209, 210, 211

E. Hmm. Let's count them again. This time, why don't you count these (tens and ones).

S. (Touching the tens) 10, 20, 30, 40, 50, 60, 70 (touching the ones) 71, 72, 73, 74.

E. How much is this (hundreds)?

S. 200.

E. Okay. How much is that altogether?

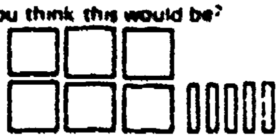
S. 200 and . .

E. I have 200, and I add this much (a ten block) more. How much is that worth?

S. 201.

b) Jane

E: Good. So how much do you think this would be?



S. (Touching the hundreds blocks) 100, 200, 300, 400, 500, 600 (touching the tens blocks) 700, 800, 900, ten hundred, eleven hundred

E. Are these (tens) worth 100?

S. I count them all together.

E. But these (tens) aren't hundreds.

S. I am counting these like tens.

E. OK. But how much would these (tens) be worth then?

S. Oh. 10, 20, 30, 40, 50 . 50 dollars.

E. How much would this (entire display) be worth altogether?

S. 600 . . . wait! It's 5 and 6.

E. But how much is it altogether? This (hundred) is 6, right?

S: Eleven hundred.

Figure 3.10 Examples of confusions in multidenominational counting.

of the different parts that make up the whole number. For example, most second- through fourth-graders we have interviewed compared numbers on the basis of the higher-value digits without reference to the lower-value positions. For example, when comparing written numerals or block displays for the numbers 472 and 427, a child would typically say 472 was larger ". . . because it has 7 tens (or 70) and the other only has 2 tens." It is interesting to note that these judgments assume that the block displays are *canonical*—that is, that they contain no more than 9 blocks of a given denomination. The assumption of canonicity disappears in the second stage of decimal knowledge, as we shall see next.

MENTAL ARITHMETIC

The most stunning displays of a compositional representation of number are in children's invented mental calculation methods. Consider the following performance by an 8-year-old, Amanda:

E: *Can you subtract 27 from 53?*
A: *34*
E: *How did you figure it out?*
A: *Well, 50 minus 20 is 30. Then take away 3 is 27 and plus 7 is 34.*

Amanda came up with the wrong answer, but by a method that clearly displayed her understanding of the compositional structure of two-digit numbers. She first decomposed each of the numbers in the problem into tens and units, and then performed the appropriate subtraction operation on the tens components. Next she proceeded to add in and subtract out the units components. She should have subtracted 7 and added 3, but instead reversed the digits. Amanda performed on other problems without this difficulty, yielding correct answers. Other children have shown similar strategies.

We have also begun to explore decimal-based mental arithmetic using the reaction-time methods that yielded initial evidence for the *min* and *choice* procedures for smaller numbers. We now have reaction-time data from 12 second- and third-grade children on a set of problems of the form 23 + 9, 35 + 2, 48 + 5. In each problem the two-digit number was presented first and fell within the 20s, 30s, or 40s decade. Each child responded to three sets of 100 such problems; the sets consisted of all possible pairings of the units digits, with the tens digits allowed to vary randomly. The problems were presented horizontally on a videoscope, and the child responded on the digit keys of a computer terminal. Time from presentation to response was recorded.

Assuming that one is going to use a mental counting procedure for solving these problems, there are two plausible possibilities that distinguish clearly between use and non-use of the decade structure:

1. Set the counter to the two-digit number, then add in the one-digit number in increments of one. Reaction time would be a function of the single-digit number (in this case, always the second number). We call this the *min of the addends* procedure. No understanding of the decade structure of the numbers is required for this procedure. However, the child does have to know how to count over the decade barrier (e.g., ". . . 29, 30, 31 . . .") and must have a units number string that extends up through several decades.

2. Decompose the two-digit number into a tens component and a ones component, then recombine the tens component with whichever of the two units quantities is larger. Set the counter to this reconstituted number and then add in the smaller units digit in increments of one. For example, for 23 + 9, the counter would be set at 29 and then incremented three times to a sum of 32. Reaction time would be a function of the smaller of the two *units* digits, so the procedure is called *min of the units*. This procedure is a simple version of the one Amanda used. It not only uses the decade structure of the numbers but behaves in accord with principles of commutativity and associativity (e.g., $23 + 9 = [20 + 3] + 9 = 20 + [3 + 9] = 20 + [9 + 3] = [20 + 9] + 3 = 29 + 3$).

We fit each of these models (along with several others that are plausible but whose use would not clearly illuminate decimal structure knowledge) to the reaction times (correct solutions only) of each of our subjects. We predicted the pattern of reaction times for a "pure" model, for a model with very fast times for doubles in the units digits, and for a model with very fast times for tens complements (i.e., pairs that add to 10, such as 3 + 7, 6 + 4, etc.). We also interviewed each child on a set of similar problems in a think-aloud format. Finally, we had reaction-time data on each child's performance on a set of single-digit addition problems. Because a purely mathematical discrimination between models is so difficult (the models themselves are highly intercorrelated), we used a combination of model fits, plausibility of the slopes (presumed counting speeds), children's think-aloud protocols, and the match between lower decade (single digits) and upper decade (two digits plus one digit) performances to tease out a story about each child's performance.

Two children, Ken and Alan, provide particularly clear illustrations of the differences between children who are in a predecimal stage of number representation and those who are clearly using a decimal representation in their mental arithmetic. Ken's reaction times on the upper decade problems were best fit by *min of the addends* ($r^2 = .761$). On the single-digit problems his data cleanly fit the *min* model, with doubles ($r^2 = .695$). The slope of the regression lines for the upper and lower decades (1.164 and .960, respectively) indicated a mental counting time of about one second per increment for both kinds of problems. This suggests that Ken was using the same basic units-counting strategy for both the single- and the two-digit problems. Ken also described the *min of the addends* counting-up procedure as his method in the think-aloud protocols.

Alan provides a contrast case. His reaction times on the upper decade problems fit best the *min of the units* model (r^2 = .847). He also showed a next-best fit for *min of the units with complements*, the only child to show a good fit to any complements model; and he showed a reassuringly poor fit to the *min of the addends* model. On the single-digit problems, his data best fit *min*, with doubles (r^2 = .831). His slopes for upper and lower decade problems were also similar (.346 for the single-digit problems; .441 for the two-digit problems), indicating a similar mental counting speed for both kinds of problems. Although this story seems very straightforward, it is also incomplete, for Alan's data also fit (although with less variance explained) other models. It seems quite likely that he was using a variety of strategies on different problems. This impression is confirmed by his interview data. He clearly described himself as using the *min of the units* strategy for some problems, but on others he described various other methods that relied on knowledge of doubles and complements. It seems reasonable to conclude that Alan was using complex representations of number relationships to generate strategies that included but were not limited to *min of the units*.

OTHER STAGE ONE TASKS

There are a number of tasks that an individual with the compositional representation of number shown in Figure 3.8 ought to be able to perform, but on which we have only impressionistic data at the present time. These include:

1. adding or subtracting 10 from any quantity more quickly than adding or subtracting other numbers (except 0 or 1, and *possibly* 2). To subtract 10 from 47, for example, an individual could enter the representation at 47 and move one step on the "tens-backward-next" link directly to 37.
2. counting up (or down) by tens from any starting number.
3. constructing mental addition and subtraction algorithms that use the ability to count by 10 from any number. For 72 − 47, for example, enter the number representation at 72; move down the 10 string four positions to 32. Move down the ones string (crossing the tens position) seven positions to 25. This strategy is related to those (such as *min of the units* and Amanda's strategies) that partition numbers and operate separately on the tens and units, but it reflects a somewhat different use of the decimal structure.

A FORMAL THEORY OF STAGE ONE KNOWLEDGE

We are able to benefit in our analysis of the development of decimal number knowledge from a computer program that simulates the performances of a 9-year-old girl, Molly, on a number of the tasks that provide the basis for inferring place-value knowledge. The program, MOLLY, matches Molly's performance

at several points before, during, and after remedial tutorial instruction aimed at establishing an understanding of the rationale for the standard, school-taught written subtraction algorithm. Prior to our instruction, Molly demonstrated the ability to perform tasks such as constructing, interpreting, and comparing block displays of two- and three-digit numbers. The knowledge structure included in the program that was used in performance of all of these tasks is shown schematically in Figure 3.11. This structure organizes conventional information about multidigit written numbers. The structure identifies columns according to their positional relationship to each other. The rightmost column is tagged as the units column, the tens column is the one that is next to the units, the hundreds is next to the tens, and so forth. Which column is being attended to can be determined by starting at the rightmost position and running through the succession of Next links. Attached to each column is a block shape (the block names are those used by Dienes, 1966, in referring to blocks such as those in Figure 3.9), a counting string, and a column value. The value specifies the amount by which a digit must be multiplied to yield the quantity represented by the digit (e.g., in the tens column, multiply by 10).

Someone who possessed this knowledge structure should be able to associate block shapes with column positions, block shapes with column values, and so on. Table 3.1 gives the number of third-grade children in one of our studies who showed reliable knowledge of each type of association at each of two interview points during the year. Since the knowledge was inferred from the method by which children solved the various problems presented, rather than by direct questioning, it was not possible to observe each child on each association in each interview. For this reason the data are given as proportions—the number of children who showed knowledge of the association over the number observed.

As can be seen, all of the children had the position-name association from the outset. That is, they could read two- and three-digit numbers aloud using the proper conventions. A position-shape association was inferred when the children constructed displays in a manner that directly matched each block shape to a digit. The children using a column-by-column match strategy typically worked on the leftmost column first and pointed to each column in succession, saying, "*n* of these." Three of our subjects worked this way successfully in their first interview, more in the second interview. All of the children we observed could apply the appropriate counting strings to block shapes as long as there was only a single block shape to be counted. When they had to switch denominations (hundreds to tens, or tens to ones), however, they had difficulty: Less than half of those observed succeeded (cf. Figure 3.10). To be counted as knowing the value of a column position, the child had to either tell us that, for example, a 9 in the tens column was "worth" 90, or select 9 tens blocks to represent that quantity. Only one child demonstrated this knowledge. Nevertheless, the children demonstrated fairly strong knowledge of the value of block shapes, as is shown in the final row of Table 3.1.

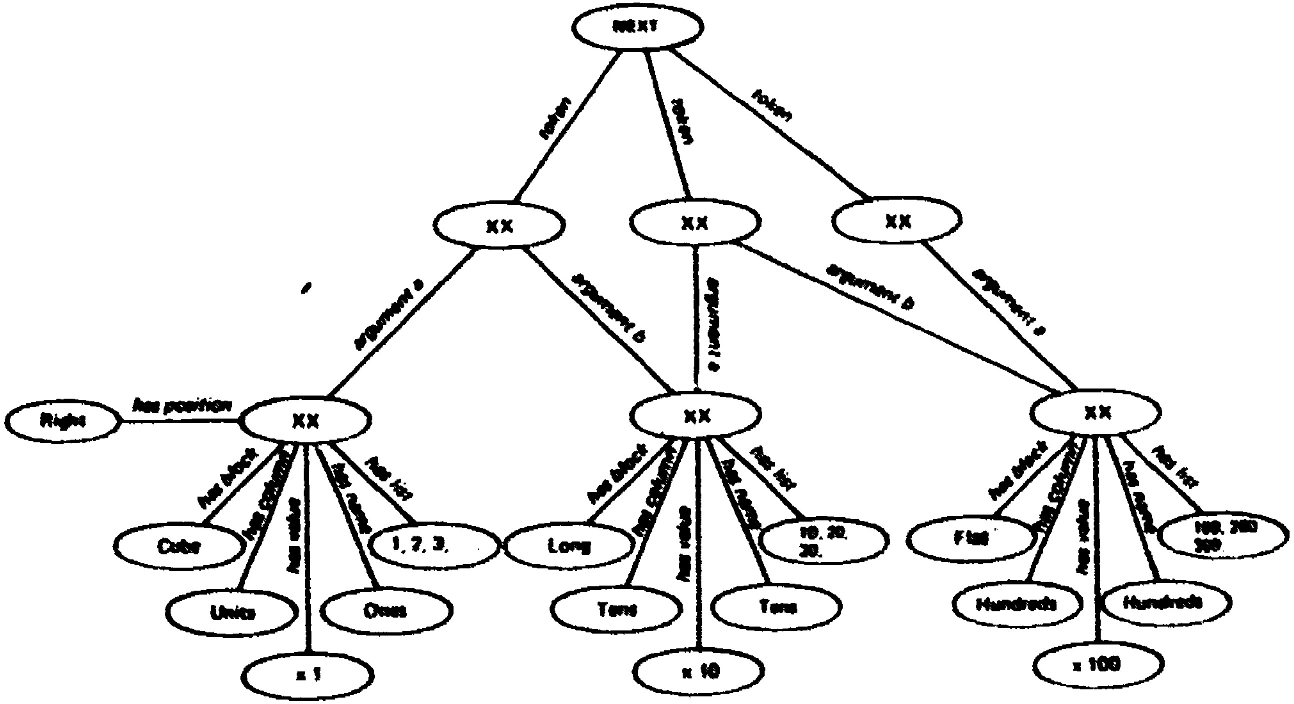


Figure 3.11 The Next structure: Knowledge about multidigit written numbers and associated blocks

Table 3.1
Proportions of Third Grade Children Displaying Knowledge of Associations between Columns, Blocks, and Values in November and February Interviews

	November	February
Column position/Column name	10/10	10/10
Column position/Block shape	3/3	6/7
Counting strings/Block shapes		
One denomination only	6/6	7/7
Two or more denominations	2/7	3/6
Column position/value	1/10	1/7
Block shape/value	7/10	9/10

Stage Two: Multiple Partitionings of Multidigit Numbers

As long as the Next structure alone is used to interpret numbers, each written number can have only one block representation: a "canonical" representation, with no more than 9 blocks per column. In this canonical display there exists a one-to-one match between the number of blocks of a particular denomination and the digit in a column in standard written notation. Insistence on the canonical form, however, means that there is no basis for carrying and borrowing—or, in block displays, for exchanges and multiple representations of a quantity. During the next stage in development, the Part–Whole schema is applied to multidigit numbers in a manner that allows multiple partitionings and thereby a variety of noncanonical representations of quantity.

MULTIPLE PARTITIONING ARRIVED AT EMPIRICALLY

At first, although children recognize that multiple representations are possible, they can construct them only through an empirical counting process. Molly's performance during the preinstructional phase of her work with us illustrates this method. Molly was asked to use Dienes blocks to subtract 29 from 47. She began by constructing the block display that matched the larger number—that is, 4 tens and 7 units. She then tried to remove 9 units and, of course, could not. The experimenter asked if she could find any way to get more units. Molly responded by putting aside all of the units blocks and one of the tens in her display, leaving just 3 tens. She counted these by tens ("10, 20, 30") and then continued counting by ones, adding in a units block with each count, up to 47. On the next subtraction problem, 54 − 37, Molly began with a noncanonical display of the top number. That is, she put out 4 tens and counted in units blocks until she reached 54, yielding a final display of 4 tens and 14 units. Molly thus appeared to have learned that certain problems will require noncanonical displays; she had

incorporated into her plan for doing block subtraction a check for whether there were more units to be removed than the canonical display would provide. However, at this stage she was able to establish the equivalence of the canonical and noncanonical displays only by the counting process that yielded the same final number in each case.

The MOLLY program provides a formally stated theory of what Molly knew and how she used her knowledge at each of several stages. To simulate the stage of performance just described, MOLLY-1 uses several procedures that call upon the Part–Whole schema described earlier for story problems. In MOLLY-1, the schema is elaborated to include a special restriction, applied to two-digit numbers, that one of the parts be a multiple of 10. To "show 47 with more ones," MOLLY-1 first applies Part–Whole in a global fashion, concluding that if the Whole is to stay the same but more ones are to be shown, there must be fewer tens. MOLLY-1 then reduces the tens pile by a single block, the smallest possible amount to remove. Next, the schema is instantiated with 47 filling the Whole slot, and 30 in one of the Part slots. The Minimum Needed node of the schema is then used to access a procedure for finding the remaining Part by adding ones blocks and counting up until 47 is reached.

Two important concepts have been added to the number representation at this stage. First, the equivalence of several partitionings has been recognized. Second, the possibility of having more than 9 of a particular block size has been admitted. This is crucial for an eventual understanding of borrowing, where—temporarily—more than 9 of a given denomination must be understood to be present, without changing the total value of the quantity. Interviews with a number of children in addition to Molly make it clear that prior to this stage the possibility of borrowing or trading to get more blocks is rejected because it will produce an "illegal" (i.e., noncanonical) display.

PRESERVATION OF QUANTITY BY EXCHANGES THAT MAINTAIN EQUIVALENCE

A complete understanding of the possibilities for multiple representation can be attributed to children only when they are no longer dependent upon counting to establish the equivalence of displays—that is, when they recognize a class of legal exchanges that will automatically preserve equivalence. Although Molly received no explicit instruction from us on this point, it was clear that after a certain amount of practice with the counting-up method of creating noncanonical displays, she came to recognize that 10-for-1 exchanges would retain the Whole quantity while changing the specific amounts in the Parts. At this point she stopped counting up and began simply to trade—that is, discard a tens block and count in 10 units, or discard a hundreds block and count in 10 tens. We have observed the same kind of performance in other children as well. Some children

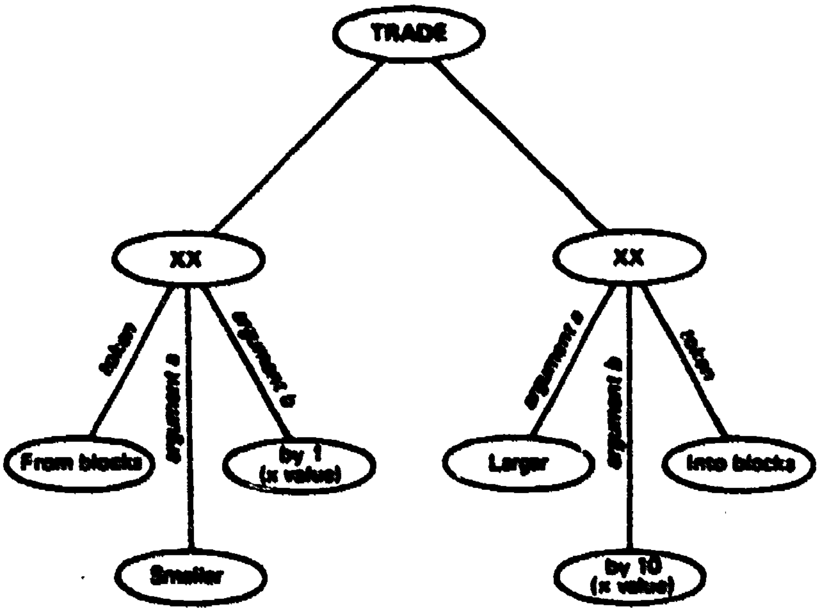


Figure 3.12 The Trade schema.

who engage in trades rather than counting up even become annoyed or amused with an experimenter who keeps asking how they know that the display still shows the same number. They indicate in various ways that they believe that if a ten-for-one trade has been made, the total quantity could not have changed.

The MOLLY-2 program provides a formal theory of Molly's knowledge at this stage. In what can be viewed as a further elaboration of the Part–Whole schema, MOLLY-2 adds to the representation for multidigit numbers an explicit 10-for-1 relationship for adjacent block sizes. This knowledge is represented by a Trade schema (Figure 3.12), which specifies a class of legal exchanges among blocks. The schema specifies that there is a "from" pile of blocks from which blocks are removed. This pile becomes smaller by one block. There is also an "into" pile of blocks that becomes larger by 10 blocks. The *value* of the blocks in the From and Into piles is established by multiplying the number of blocks removed or added by the value of the block shape (as specified in the Next structure and a separate Value schema that is also part of the program). Thus, when trades are made between adjacent block sizes, the schema specifies that the Into and the From values will be the same, even though the number of physical objects present has changed. Applied as an elaboration of the Part–Whole schema, the Trade schema allows MOLLY-2 to construct alternative partitionings of a quantity without having to count up from one of the parts.

Stage Three: Application of Part–Whole to Written Arithmetic

I turn now to children's written arithmetic—in particular, to how the elaborated Part–Whole schema is eventually applied to the interpretation of the con-

ventions of written calculation. There is abundant evidence now available that many children learn rules for the written algorithms of subtraction and addition without linking these rules to the kind of knowledge about place value and number that I have described here. What they seem to learn is a procedure for identifying columns, operating on them, making marks (writing in little 1's, crossing out and rewriting numbers, etc.), but not a rationale that makes the procedure sensible. Brown and Burton (1978) have demonstrated that when children make errors in written arithmetic—particularly subtraction—the errors are often the result not of random mistakes, but of the systematic application of wrong ("buggy") algorithms. Figure 3.13 describes and illustrates some of the most common subtraction bugs. Elsewhere (Resnick, 1981) I have analyzed a number of the Brown and Burton bugs to show that they typically follow rules of syntax, or procedure, while ignoring or contravening the "semantics" of exchange—that is, the principles embodied in the Part–Whole, Trade, and Value schemata described here. For example, in the bug called Borrow-Across-Zero the child follows a rule specifying the need for a written-in little 1 and a crossed-out and decremented number to its left. The syntax of subtraction is largely respected. However, the semantics of exchange is violated, for the child has in fact borrowed 100 but added back only 10—thus failing to conserve the original quantity.

Brown and VanLehn (1980, 1982; VanLehn, Chapter 5) have developed a theory intended to account for the process by which buggy algorithms are invented. The theory assumes that the correct algorithm has been learned but is incomplete for certain problems, either because an incomplete algorithm was taught or because certain steps have been forgotten. When these problems—which most often contain zeros in the top number—are encountered, the attempt to apply the learned algorithm creates an impasse. The child attempts to cope with the impasse by "repairing" the learned algorithm. The repairs proceed in a "generate-and-test" mode that is shared with many other problem-solving process theories (e.g., Newell & Simon, 1972). First, a repair is generated from a very limited list of potential repairs. The list includes moving into the next column to perform an action (this would produce the Borrow-Across-Zero bug), skipping an action, copying a number, and the like. Once generated, a repair is tested against a set of "critics" that specify certain constraints that a subtraction algorithm must obey. These include rules such as acting at least once on each column, showing decrement and increment marks, and not writing more than one digit in each answer column. There is nothing in either the critic list or the repair generation list that refers to what I have been developing in this chapter as the "meaning" of decimal numbers. There is no critic that specifies that the original Whole quantity must be preserved, nor is there anything in the repair or critic lists that even identifies the value of the borrow and increment marks. The theory thus describes an almost wholly syntactic set of bug-generating processes.

Given this characterization of the origin of buggy arithmetic, it can be

1. Smaller-From-Larger. The student subtracts the smaller digit in a column from the larger digit regardless of which one is on top.

$$\begin{array}{r} 326 \\ -117 \\ \hline 211 \end{array} \qquad \begin{array}{r} 542 \\ -389 \\ \hline 247 \end{array}$$

2. Borrow-From-Zero. When borrowing from a column whose top digit is 0, the student writes 9 but does not continue borrowing from the column to the left of the 0.

$$\begin{array}{r} 602 \\ -437 \\ \hline 265 \end{array} \qquad \begin{array}{r} 802 \\ -396 \\ \hline 506 \end{array}$$

3. Borrow-Across-Zero. When the student needs to borrow from a column whose top digit is 0, he skips that column and borrows from the next one. (Note: this bug must be combined with either bug 5 or bug 6).

$$\begin{array}{r} 702 \\ -327 \\ \hline 335 \end{array} \qquad \begin{array}{r} 904 \\ -456 \\ \hline 309 \end{array}$$

4. Stops-Borrow-At-Zero. The student fails to decrement 0, although he adds 10 correctly to the top digit of the active column. (Note: this bug must be combined with either bug 5 or bug 6).

$$\begin{array}{r} 703 \\ -678 \\ \hline 175 \end{array} \qquad \begin{array}{r} 604 \\ -387 \\ \hline 307 \end{array}$$

5. 0 – N = N. Whenever there is 0 on top, the digit on the bottom is written as the answer.

$$\begin{array}{r} 709 \\ -352 \\ \hline 457 \end{array} \qquad \begin{array}{r} 6008 \\ -327 \\ \hline 6321 \end{array}$$

6. 0 – N = 0. Whenever there is 0 on top, 0 is written as the answer.

$$\begin{array}{r} 804 \\ -462 \\ \hline 402 \end{array} \qquad \begin{array}{r} 3050 \\ -621 \\ \hline 3030 \end{array}$$

7. N – 0 = 0. Whenever there is 0 on the bottom, 0 is written as the answer

$$\begin{array}{r} 976 \\ -302 \\ \hline 604 \end{array} \qquad \begin{array}{r} 856 \\ -409 \\ \hline 407 \end{array}$$

8. Don't-Decrement-Zero. When borrowing from a column in which the top digit is 0, the student rewrites the 0 as 10, but does not change the 10 to 9 when incrementing the active column.

$$\begin{array}{r} 702 \\ -368 \\ \hline 344 \end{array} \qquad \begin{array}{r} 105 \\ -9 \\ \hline 1106 \end{array}$$

9. Zero-Instead-Of-Borrow. The student writes 0 as the answer in any column in which the bottom digit is larger than the top.

$$\begin{array}{r} 326 \\ -117 \\ \hline 210 \end{array} \qquad \begin{array}{r} 542 \\ -389 \\ \hline 200 \end{array}$$

10. Borrow-From-Bottom-Instead-Of-Zero. If the top digit in the column being borrowed from is 0, the student borrows from the bottom digit instead (Note: this bug must be combined with either bug 5 or bug 6).

$$\begin{array}{r} 702 \\ -378 \\ \hline 454 \end{array} \qquad \begin{array}{r} 508 \\ -489 \\ \hline 109 \end{array}$$

Figure 3.13 Descriptions and examples of Brown and Burton's (1978) common subtraction bugs. (Adapted from Resnick, 1982. Copyright 1982 by Lawrence Erlbaum Associates. Reprinted by permission.)

argued that one of the important tasks of primary school arithmetic learning is the development of knowledge structures that provide a "semantic justification" for procedures of written borrowing and carrying. As we have seen earlier in this discussion, there is evidence that children have or can relatively easily acquire substantial semantic knowledge—in the form of Part–Whole and Trade schemata and associated procedures—applied to concrete representations of number. It

therefore seems likely that a useful method for assisting children in the development of a semantic interpretation of written arithmetic would be to call their attention to correspondences between the steps in written arithmetic and the performance of addition and subtraction with concrete materials (cf. Dienes, 1966). In an earlier work (Resnick, 1981) I described one method for doing this, via what was termed *mapping instruction*. In this instruction the child is required to perform the same problem in blocks and in writing, alternating steps between the two. Under these conditions the written notations can be construed as a

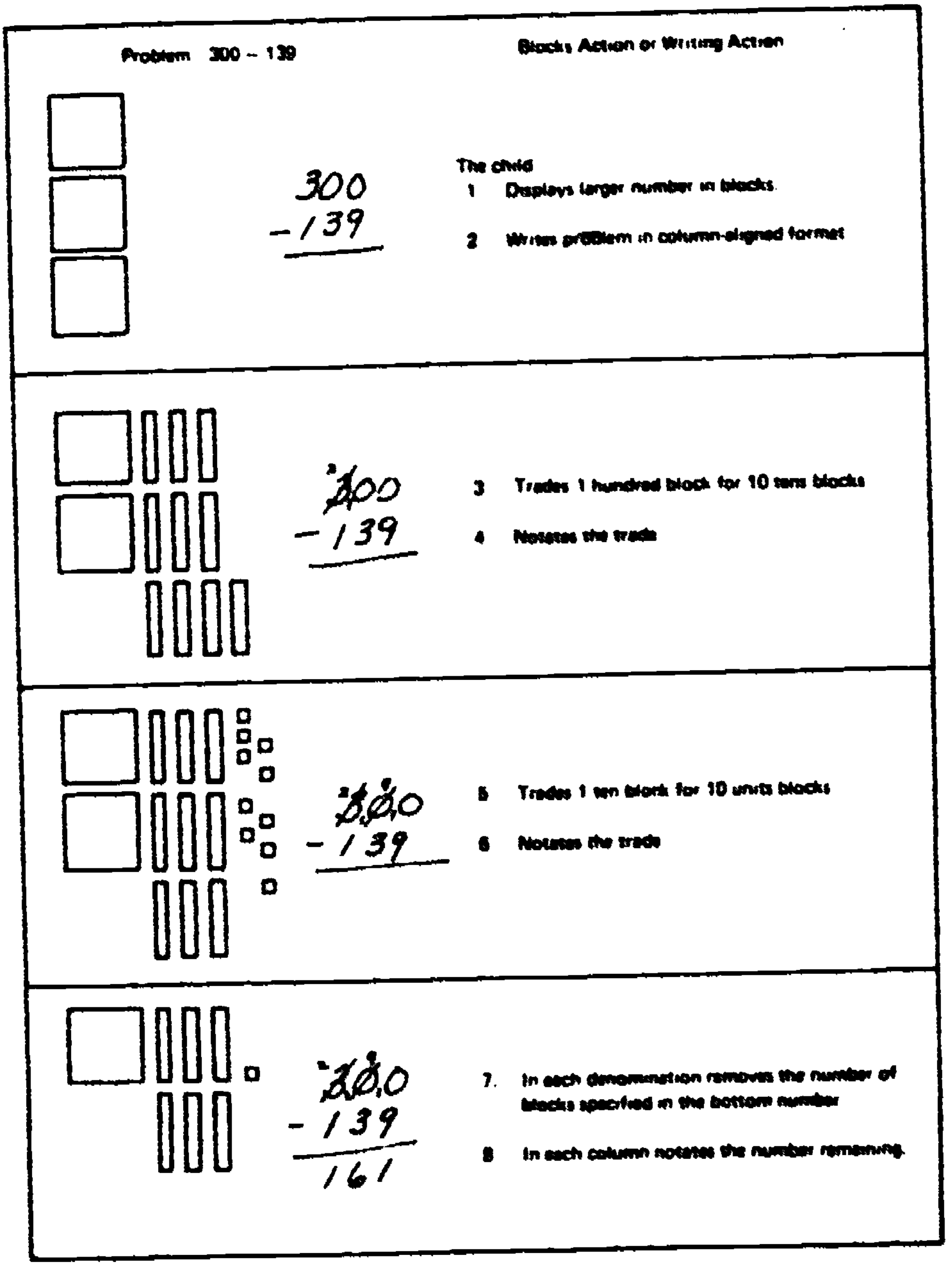


Figure 3.14 Outline of mapping instruction for subtraction. (From Resnick, 1982. Copyright 1982 by Lawrence Erlbaum Associates. Reprinted by permission.)

"record" of actions on the blocks. Figure 3.14 summarizes the process for a subtraction problem.

Mapping instruction has been successfully used with several children who had buggy subtraction algorithms. Not only did their bugs disappear, but the children demonstrated that they had acquired an understanding of the semantics of the written algorithm. Once again, Molly's performance and our simulation of it provide both a clear example of typical behavior and a theoretical account of the mental processes involved.

VALUE OF CARRY AND BORROW MARKS

We have seen that rather early in their development children can recognize the values of digits in various columns of standard notation, using the Next structure only. There is evidence in our data, however, that this ability to assign value does not extend to the notations made in the course of carrying and borrowing. In one of our studies, third-grade children were asked to tell us the value of the carry and borrow digits in written addition and subtraction. In virtually every case they simply named the digit rather than its actual value. For example, when they were shown the solved problem in Figure 3.15, the little 1 at *a* was assigned a value of 1 instead of 10, and the little 1 at *b* was assigned a value of 1 instead of 100. When asked to select the block(s) that would represent these 1 marks, the children typically selected a single units block. By contrast, after instruction Molly and others who had been taught via mapping assigned a value of 10 to the 1 at *a* and 100 to the 1 at *b*, and selected blocks accordingly.

EXPLAINING THE WRITTEN BORROWING ALGORITHM

Molly's most stunning display of understanding written borrowing came in a follow-up interview about four weeks after instruction. During this time she had had no direct instruction on subtraction. When asked to do problems in writing in this follow-up interview, Molly did not use exactly the procedure she had learned from us. That is, on problems with 0 in the top number, she did not begin by decrementing in the hundreds column and changing the 0 in the tens

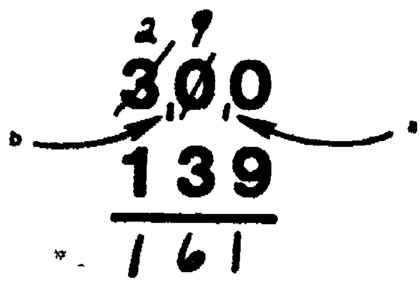
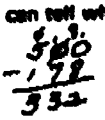


Figure 3.15 Solved problem showing carry and borrow marks

a) E I want to show you some problems that someone else did and see if you can tell whether this person did it correctly or not. This is the problem:



See if you can check that, and check all the steps and make sure it was done correctly. If you see something wrong, tell me what's wrong.

S She left it a 10—kept it a 10.

E What should she have done?

S Made it a 9.

E Why is that?

S To take 90 tens from here (hundreds) and then the other 10 would go there (the ones)

E How many do you take from here (hundreds) altogether?

S A hundred.

b) E OK, so how do you write that?

S You put 10 there (13) and 9 there (in tens), which is 90, and 9 there (hundreds), which is 900. (Writes )

E OK So where are the 10 hundreds in the writing?

S 100 is right here (points to top digits in the ones column and tens column) and 900 is right here (points to the hundreds column).

Figure 3.16 Two extracts of Molly's explanations

column to 10, then decrementing this 10 to produce 9 as part of the exchange into the units column. Instead, she used the "school algorithm," going right to left and changing each 0 directly to 9.

This algorithm cannot be directly mapped onto blocks, and thus one cannot explain why it works by simply describing exchanges as if they had been done with blocks. Thus, any justification Molly was able to offer for her written work would have to depend on her schematic knowledge. Figure 3.16 gives two extracts of Molly's explanations. In the first case Molly was asked to check another child's work. She knew the 10 in the tens column should be changed to 9, but she did not justify this as the outcome of a trade. Instead, she gave an explanation in terms of the values of the decrement and increment marks (9 tens in the tens column plus 1 ten in the units column), with the clear implication that a whole-preserving exchange had been made (otherwise she would not have sought the "other ten"). In the second extract, Molly shows even more clearly that she was searching for parts to make up the 1000 that she recognized had been borrowed in the course of decrementing the thousands column.

MOLLY-3 provides a theory of how these explanations were constructed. To construct analogous explanations, MOLLY-3 uses an Exchange schema (Figure 3.17) that develops by interpreting borrowing as an analog of trading. The Trade and the Borrow portions of the Exchange schema have analogous elements. As a result, for written borrowing there is a From column that gets smaller by 1 and an Into column that gets larger by 10. The *values* of these

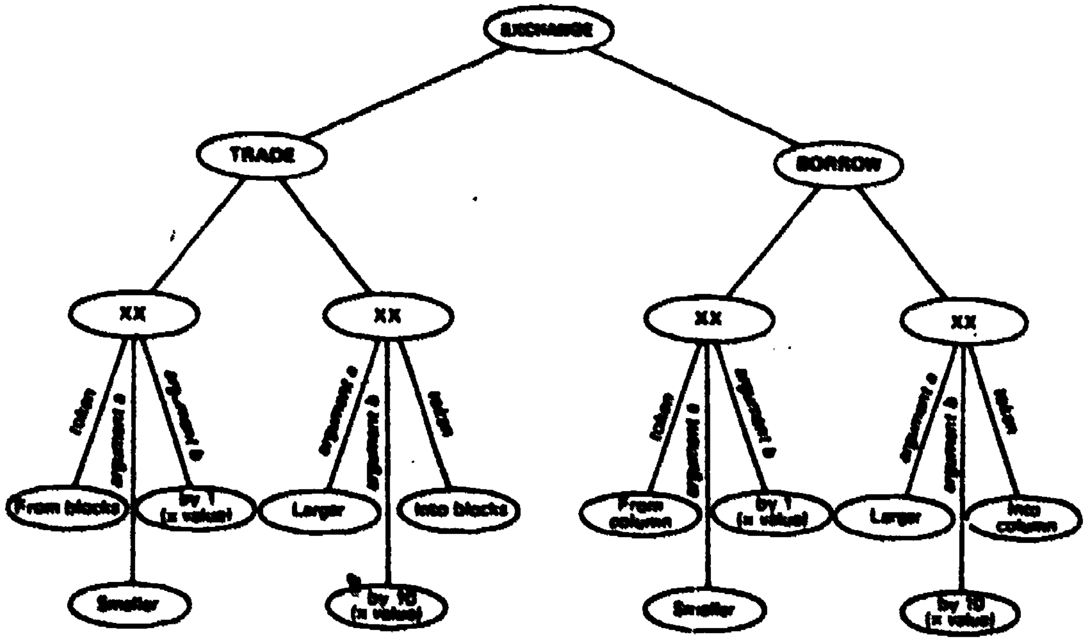


Figure 3.17 The Exchange schema.

decrements and increments are, as in the case of trading, determined by multiplying by the column value. In the units column, the increase of 10 in the Into column is multiplied by 1; but in the tens column it is multiplied by 10 to yield a value of 100. As a result, when interpreted under control of the Exchange schema, the increment marks would be represented by tens or hundreds blocks, never by unit cubes. The effect of having the Exchange schema is to allow MOLLY-3 to interpret borrowing as it had trading: as an exchange among parts that maintains the value of the whole.

MOLLY-3 uses its newly constructed Exchange schema to construct explanations for the standard school borrowing algorithm that parallel those of Molly. For example, for the problem 403 − 275, MOLLY-3 handles several questions about increments and decrements as follows: It keeps track of its actions by building a temporary Changes structure that specifies old and new values in particular columns. The Changes structure also records whether the new value is larger or smaller than the original. Faced with the question, *Where did the 13 in the units column come from?*, the program examines its current Changes structure, searching for a 13 as a *new value* in the units column. Finding this, it can determine that the 13 is *larger* than the original value for that column. Now it looks for a place in its knowledge where *larger* is linked with a column *into* which something is added. It finds the Borrow schema. It instantiates this schema, with the units column as the Into column. It can then "read out" the answer from the instantiated schema as: *It comes from borrowing 1 ten from the tens column for the units column.*

Now given the question, *Where is the 100 you borrowed from the hundreds column?*, MOLLY-3 uses its Changes structure to determine that the hundreds column is *smaller*. As a result, it searches for a structure in its own knowledge base in which a column is made *smaller* by taking something *from* it. This leads it to the Borrow schema, which it activates and tries to instantiate. It fills the From column with the hundreds column, and it knows this column has gotten smaller by one times the column value (of 1000). It must now fill the slots on the Into side. To do this it tries at first to find a column made *larger* by 10 times a column value of 10, but it cannot find such a column in the written notation. Instead, it finds a value of 90 shown in the tens column. At this point it calls on the Part–Whole schema, sets the Whole equal to 100 and Part A equal to 90. From this it can determine that Part B must equal 10. Now it inspects the written notation again, looking for a column that shows an increment with a value of 10. It is able to find this in the units column of the written notation. As a result it can conclude that: *The 100 from the hundreds column has been made into the 90 in the tens column plus the 10 in the units column*. MOLLY-3 can answer the analogous question for borrowing across two zeros (for example, when 2003 is the top number in a problem) by iterating through the Part–Whole schema twice, first setting the Whole slot equal to 1000 and Part A to 900, then setting the Whole to 100 and Part A to 90. It then answers: *The 1000 from the thousands column has been made into the 900 in the hundreds column, plus the 90 in the tens column, plus the 10 in the units column.*

CONCLUSION

Other topics in mathematics (multiplication, division, fractions), of course, will have been introduced by the end of the early school years and will have induced changes in representation not considered here. Nor can it be expected that all children by the end of primary school will have achieved the level of understanding represented by Molly. Yet such understanding is certainly an important goal of early instruction in place value. Thus, it seems a suitable point at which to conclude this account of the cognitive development that accompanies early school arithmetic learning. What general conclusions about the nature of number understanding and its development can be drawn from this account?

The Centrality of the Part–Whole Schema in Number Understanding

First, it seems clear that a reasonable account of the knowledge underlying changing mathematics competence can be given in terms of a few schemata and

their successive elaborations. As we have seen, the Part–Whole schema plays a central role. Although I have not attempted here to explain the origin of the Part–Whole schema, it seems likely that it arises in connection with various real-life situations in which partitions must be made but no exact quantification is required. Such situations are easy to imagine in the life of the young child. For example, a hole in an otherwise complete puzzle means that a *part* is missing; food is shared with the recognition that the individual portions together represent *all* (the *whole*) that is available; or a child gives *some* (but not *all*) of her candy to her brother.

I have pointed to evidence that Part–Whole in this primitive form is available to children before school begins. I have also suggested that its systematic application to *quantity* characterizes the early years of school. A first elaboration of the basic Part–Whole schema, in this view, is its attachment to procedures for counting up (the procedures attached to the Minimum Needed node) and taking away (the procedures attached to the Maximum Exceeded node). These procedures, which are based on the units number string, produce a quantitative interpretation of Part–Whole. The schema in turn allows numbers to be interpreted *both* as positions on the mental number line *and, simultaneously*, as compositions of other numbers. This interpretation of number appears to underlie both story-problem solution and the invented mental arithmetic procedures for small numbers that characterize the earliest school years.

Further elaboration of the Part–Whole schema appears to characterize subsequent development of an understanding of the place-value system of notation and the calculation procedures based on it. Children apparently find it easy to place a special restriction on Part–Whole such that one of the parts must always be a multiple of 10. This initial elaboration generates an interpretation of multidigit numbers as compositions of units, tens, hundreds, and so on. This in turn permits invention of several quite elegant mental calculation shortcuts. However, further elaborations—those specified in the Trade and later the more abstract Exchange schemata—are required before multiple partitionings of quantity can be recognized and the rules of written arithmetic interpreted. Since Trade and Exchange are always called upon by Part–Whole, it seems reasonable to view them as elaborations of the more general schema for partitioning quantity.

Microstages in Development

Many readers will have noted parallels between the analysis offered here and interpretations of the number concept proposed by Piaget and others working in the Genevan tradition. Indeed, this analysis shares two central emphases with the Piagetian view: (*a*) an emphasis on part–whole (class inclusion, for Piaget) relationships as a defining characteristic of number understanding, and (*b*) the

proposal that ordinal (counting) and cardinal (class inclusion or part–whole) relationships must be combined in the course of constructing the concept of number.

It is especially pleasing to have arrived at this convergence because the present analysis was conducted quite independently of Piaget's work. I did not set out to either support or disconfirm Piaget's theory of number understanding but rather to build a plausible account, from a current cognitive science point of view, of what number knowledge must underlie the various arithmetic performances observed in young school children. In doing this, I drew on formal theoretical analyses that worked *from* task performances *to* the kind of knowledge children "must have" in order to engage in the performances observed. This effort to build a theory of understanding on the basis of detailed analyses of procedures used in performing tasks is quite different from the Piagetian method of hypothesizing a mental structure and then seeking tasks that might reveal its presence or absence. One might well characterize the methods used here as more bottom up than those of Piaget.

One result of these more bottom-up task- and performance-driven methods is that we are able to detect—indeed, are forced to recognize—relatively small changes in cognitive structures. In a sense, we have been able to produce a *microstage* theory for number understanding, a theory that specifies many small changes in number representation and schematic interpretation of number in a period of development for which the Piagetian analysis recognized only the *macrostages* of preoperativity and concrete operativity. This enriched theory of changes in number knowledge is of clear importance to those concerned with instruction, for it specifies "what to teach" at successive stages of learning or development. The microstages of understanding developed here also permit us to give a more precise psychological interpretation to certain key mathematical concepts than has heretofore been possible.

An Interpretation of Cardinality

One example of such interpretation is the one that is now possible for the development of an understanding of cardinality. Gelman and Gallistel (1978) included in their principles of counting a cardinality principle, which specifies that the final count word reached when a set of objects is being enumerated is the total number in the set—that is, the set's cardinality. For the preschool child, who has not yet come to interpret quantity in terms of a fully developed Part–Whole schema, this is the only meaning of cardinality available. This criterion of understanding cardinality has been criticized (e.g., Bessot & Comiti, 1981) as too weak and in particular as not reflecting the Piagetian definition of cardinality. We can now see that a higher stage of cardinality understanding can

be recognized in the child's subsequent application of the Part–Whole schema to number. Although a primitive form of partitioning is clearly present in early counting behavior (this is what is required to keep counted and not-yet-counted objects separate), the Part–Whole schema used later in solving story problems yields the understanding that a total (whole) quantity remains the same even under variant partitionings.

The meaning of cardinality is further elaborated when the place-value schemata outlined here are acquired. When the Part–Whole schema with the multiple-of-10 restriction is applied to two-digit numbers, the amount represented by the number becomes subject to multiple partitioning without a change in quantity. This is exactly parallel to the new understanding of cardinality for smaller numbers that was achieved when the Part–Whole schema was applied to them. Without application of the Part–Whole schema, the cardinality of a number resides in the specific display set and the number attached to it through legal counting procedures. With Part–Whole, cardinality resides in the total quantity, no matter how it is displayed or partitioned.

The Trade and Exchange stages of multidigit number representation show yet a higher level of understanding of cardinality. At these stages it is recognized that cardinality is not altered by a specified set of legal exchanges. An analogy can be drawn with the earlier recognition of quantity as unchanged under various physical transformations (such as spreading out a display of objects—the classic Piagetian test of conservation). However, the transformations produced under control of the Trade schema do in fact involve a change in the actual number of *objects* present. Thus, recognition that the value of the total quantity remains unchanged requires a level of abstraction concerning the nature of cardinality that was not required for earlier stages of understanding.

Procedural Knowledge and Understanding

An important characteristic of the account of number development offered here is the close link between procedural skill and understanding. It has been characteristic of many past efforts to promote understanding of mathematics to speak as if understanding and procedural skill were somehow incompatible. Wertheimer (1945/1959), for example, in pressing for structural understanding as the goal of education, attacked the teaching of algorithms and other aspects of "mindless drill." Piaget, too, was largely disinterested in procedural skills, despite the role that "reflective abstraction"—the process of reflecting on one's own procedures to draw out principles—plays in his theory of development (Piaget, 1967/1971). Many educators inspired by Piaget's emphasis on understanding have actively argued against any kind of procedural emphasis in mathematics instruction.

The present analyses, by contrast, suggest that procedural skill often underlies understanding. For example, the account proposed here for the invention of the *min* and *choice* calculation procedures suggests that inventions reflecting an understanding of number can come about only when procedures become well enough established that their results can be inspected and compared. Similarly, children apparently learn about the decade structure of the number system through what must be, at first, rather "mindless" repetition of conventional counting strings.

We do not yet have a full theory to propose about exactly *how* practice in counting and other arithmetic procedures interacts with existing schematic knowledge to produce new levels of understanding. Nevertheless, it already seems clear that a detailed theory of how new levels of number understanding are achieved will reveal active interplay between schematic and procedural knowledge.

REFERENCES

Anderson, J. R. (Ed.) *Cognitive skills and their acquisition*. Hillsdale, N.J.: Lawrence Erlbaum Associates, 1981.

Ashcraft, M. H., & Battaglia, J. Cognitive arithmetic: Evidence for retrieval and decision processes in mental addition. *Journal of Experimental Psychology: Human Learning and Memory*, 1978, 4(5), 527–538.

Ashcraft, M. H., & Fierman, B. A. Mental addition in third, fourth, and sixth graders. *Journal of Experimental Child Psychology*, 1982, *33*(2), 216–234.

Bessot, A., & Comiti, C. Etude du fonctionnement de certaines propriétés de la suite des nombres dan le domaine numerique [1,30] chez des éleves de fin de première année de l' école obligatoire en France. *Proceedings of the Fifth Conference of the Internationl Group for the Psychology of Mathematics Education*. Grenoble, France, July, 1981. Available from Dean Robert Karplus, College of Education, University of California, Berkeley, CA 94720.

Briars, D. J., & Larkin, J. H. *An integrated model of skill in solving elementary word problems*. Paper presented at the annual meeting of the Society for Research in Child Development, Boston, April 1981.

Brown, J. S., & Burton, R. R. Diagnostic models for procedural bugs in basic mathematical skills. *Cognitive Science*, 1978, 2 (2), 155–192.

Brown, J. S., & VanLehn, K. Repair theory: A generative theory of bugs in procedural skills. *Cognitive Science*, 1980 4(4), 379–426.

Brown, J. S., & VanLehn, K. Toward a generative theory of bugs in procedural skills. In T. Carpenter, J. Moser, & T. Romberg (Eds.), *Addition and subtraction: A cognitive perspective*. Hillsdale, N.J.: Lawrence Erlbaum Associates, 1982.

Carpenter, T., & Moser, J. The development of addition and subtraction problem-solving skills. In T. Carpenter, J. Moser, & T. Romberg (Eds.), *Addition and subtraction: A cognitive perspective*. Hillsdale, N.J.: Lawrence Erlbaum Associates, 1982.

Carpenter, T. P., Hiebert, J., & Moser, J. M. The effect of problem structure on first graders' initial solution processes for simple addition and subtraction problems. *Journal for Research in Mathematics Education*, 1981, *12*(1), 27–39.

Comiti, C. Les premières acquisitions de la notion de nombre par l'enfant. *Educational Studies in Mathematics*, 1981, *11*, 301–318.

Dienes, Z. P. *Mathematics in the primary school*. London: MacMillan, 1966.

Donaldson, M., & Balfour, G. Less is more: A study of language comprehension in children. *British Journal of Psychology*, 1968, *59*, 461–471.

Fuson, K. C. The counting-on solution procedure: Analysis and empirical results. In T. Carpenter, J. Moser, & T. Romberg (Eds.), *Addition and subtraction: A cognitive perspective*. Hillsdale, N.J.: Lawrence Erlbaum, Associates, 1982.

Fuson, K. C. & Mierkiewicz, D. B. *A detailed analysis of the act of counting*. Paper presented at the annual meeting of the American Education Research Association, Boston, April 1980.

Fuson, K. C., Richards, J., & Briars, D. J. The acquisition and elaboration of the number word sequence, In C. Brainerd (Ed.), *Progress in logical development: Children's logical and mathematical cognition* (Vol. 1). New York: Springer-Verlag, 1982.

Gelman, R. Logical capacity of very young children: Number invariance rules. *Child Development*, 1972, *43*, 75–90.

Gelman, R. & Gallistel, C. R. *The child's understanding of number*. Cambridge Mass: Harvard University Press, 1978.

Ginsburg, H. *Children's arithmetic: The learning process*. New York: D. Van Nostrand, 1977.

Greeno, J. G. Riley, M. S., & Gelman R. *Young children's counting and understanding* Paper presented at the annual meeting of the Psychonomic Society, San Antonio November 1978.

Groen, G. J., & Parkman, J. M. A chronometric analysis of simple addition. *Psychological Review*, 1972, *79*(4), 329–343.

Groen, G. J., & Poll, M. Subtraction and the solution of open sentence problems. *Journal of Experimental Child Psychology*, 1973, *16*, 292–302.

Groen, G. J., & Resnick, L. B. Can preschool children invent addition algorithms? *Journal of Educational Psychology*, 1977, *69*, 645–652.

Houlihan, D. M., & Ginsburg, H. P. The addition methods of first- and second-grade children. *Journal for Research in Mathematics Education*, 1981, *12*(2), 95–106.

Inhelder, B., & Piaget, J. *The early growth of logic in the child*. New York: Norton, 1969. (Original English translation 1964).

Klahr, D., & Wallace, J. G. *Cognitive development: An information-processing view*. Hillsdale, N.J.: Lawrence Erlbaum Associates, 1976.

Lindvall, C. M., & Gibbons-Ibarra, C. G *A clinical investigation of the difficulties evidenced by kindergarten children in developing "models" for the solution of arithmetic story problems*. Paper presented at the annual meeting of the American Educational Research Association, Boston, April 1980.

Markman, E. M., & Siebert, J. Classes and collections: Internal organization and resulting holistic properties. *Cognitive Psychology*, 1976, *8*, 561–577.

Neches, R. *Models of heuristic procedure modification*. Unpublished doctoral dissertation. Carnegie-Mellon University, Department of Psychology, Pittsburgh, Pa., 1981.

Nesher, P. Levels of description in the analysis of addition and subtraction word problems. In T Carpenter, J. Moser, & T. Romberg (Eds.), *Addition and subtraction: A cognitive perspective*. Hillsdale, N.J.: Lawrence Erlbaum Associates, 1982.

Nesher, P., & Teubal, E. Verbal cues as an interfering factor in verbal problem solving. *Educational Studies in Mathematics*, 1975, *6*, 41–51.

Newell, A., & Simon, H. A. *Human problem solving*. Englewood Cliffs, NJ. Prentice-Hall, 1972.

Piaget, J. *The child's conception of number*. New York: Norton, 1965. (Originally published 1941)

Piaget, J. *Biology and knowledge*. Chicago: University of Chicago Press, 1971. (Originally published 1967)

Potts, G. P., Banks, W. P., Kosslyn, S. M., Moyer, R. S., Riley, C. A., & Smith, K. H. Encoding and retrieval in comparative judgments. In J. N. Castellan (Ed.), *Cognitive theory* (Vol. 3). Hillsdale, NJ: Lawrence Erlbaum Associates, 1979.

Resnick, L. B. Syntax and semantics in learning to subtract. In T. Carpenter, J. Moser, & T. Romberg (Eds.), *Addition and subtraction: A cognitive perspective*. Hillsdale, NJ: Erlbaum, 1982.

Resnick, L. B., Greeno, J. G., & Rowland, J. *MOLLY: A model of learning from mapping instruction*. Unpublished manuscript, University of Pittsburgh, Learning Research and Development Center, Pittsburgh, PA, 1980.

Schaeffer, B., Eggleston, V. H., & Scott, J. L. Number development in young children. *Cognitive Psychology*, 1974, *6*, 357–379.

Sekuler, R., & Mierkiewicz, D. Children's judgment of numerical inequality. *Child Development*, 1977, *48*, 630–633.

Siegler, R. S., & Robinson, M. The development of numerical understanding. In H. W. Reese & L. P. Lipsitt (Eds.), *Advances in child development and behavior* (Vol. 16). New York: Academic Press, 1982.

Smith, L. B., & Kemler, D. G. Levels of experienced dimensionality in children and adults. *Cognitive Psychology*, 1978, *10*, 502–532.

Steffe, L., Thompson, P., & Richards, J. Children's counting and arithmetical problem solving. In T. Carpenter, J. Moser, & T. Romberg (Eds.), *Addition and subtraction: A cognitive perspective*. Hillsdale, NJ: Erlbaum, 1982.

Svenson, O., & Broquist, S. Strategies for solving simple addition problems: A comparison of normal and subnormal children. *Scandinavian Journal of Psychology*, 1975, *16*, 143–151.

Svenson, O., & Hedenborg, M. L. Strategies used by children when solving simple subtractions. *Acta Psychologica*, 1979, *43*, 1–13.

Svenson, O., Hedenborg, M. L., & Lingman, L. On children's heuristics for solving simple additions. *Scandinavian Journal of Educational Research*, 1976, *20*, 161–173.

Svenson, O., & Sjoberg, K. Solving simple subtractions during the first three school years. *Journal of Experimental Education*, in press.

Vergnaud, G. A classification of cognitive tasks and operations of thought involved in addition and subtraction problems. In T. Carpenter, J. Moser, & T. Romberg (Eds.), *Addition and subtraction: A cognitive perspective*. Hillsdale, N.J.: Lawrence Erlbaum Associates, 1982.

Wertheimer, M. *Productive thinking* (Enlarged ed.). New York: Harper & Row, 1959. (Originally published in 1945.)

Woods, S. S., Resnick, L. B., & Groen, G. J. An experimental test of five process models for subtraction. *Journal of Educational Psychology*, 1975, 67(1), 17–21.

DOCUMENT RESUME

ED 251 328 SE 045 292

AUTHOR	Resnick, Lauren B.
TITLE	A Developmental Theory of Number Understanding.
INSTITUTION	Pittsburgh Univ., Pa. Learning Research and Development Center.
SPONS AGENCY	Nat[illegible]nal Inst. of Education (ED), Washington, DC.; Na[illegible]al Science Foundation, Washington, D.C.
REPORT NO	LRD[illegible].984/38
PUB DATE	84
NOTE	46p.
PUB TYPE	Reports - Research/Technical (143)
EDRS PRICE	MF01/PC02 Plus Postage.
DESCRIPTORS	Arithmetic; *Cognitive Processes; Computation; Educational Research; Elementary Education; *Elementary School Mathematics; Error Patterns; *Learning Theories; Mathematical Models; *Mathematics Instruction; *Number Concepts; Primary Education; Problem Solving; *Psychological Studies
IDENTIFIERS	*Mathematics Education Research

ABSTRACT

Research on the psychological processes involved in early school arithmetic has now accumulated sufficiently to make it possible to construct a coherent account of the changing nature of the child's understanding of number during the early school years. This monograph presents an account of how number concepts are extended and elaborated as a result of formal instruction. A theory of number representation is outlined for three broad periods of development: (1) the preschool period, during which counting and quantity comparison competencies of young children provide the main basis for inferring number representation; (2) the early primary period, during which children's invention of sophisticated mental computational procedures and the mastery of certain forms of story problems point to two important expansions of the number concept; and (3) the later primary period, during which the representation of number is modified to reflect knowledge of the decimal structure of the counting and notational systems. (MNS)